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DIRECTORATE OF DISTANCE EDUCATION

III - SEMESTER

M.Sc.(MATHEMATICS)

311 31

DIFFERENTIAL GEOMETRY

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DIFFERENTIAL GEOMETRY

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BLOCK-I

UNIT-I SPACE CURVES AND SURFACES

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Structure

1.1 INTRODUCTION

In this chapter on space curves, first we shall specify a space curve as the intersection of two surfaces. Then we shall explain how we shall arrive at a unique parametric representation of a point on the space curve and also give a precise definition of a space curve in E_3 as set of points associated with an equivalence class of regular parametric representations. With the help of this parametric representation, we shall define tangent, normal and binormal at a point leading to the moving triad(t,n,b) and their associated tangent, normal and rectifying planes. Since the triad(t,n,b) at a point is moving continuously as P varies over the curve,we are interested to know the arc-ratae of rotation of t, n, b. This leads to the well-known formulae of Serret – Frenet. The we shall establish the conditions for the contact of curves and surfaces leading to the definitions of osculating circle and osculating sphere at a point on the space curve and also the evolute an involutes. Before concluding this unit, we shall explain what is meant by intrinsic equations of space curves and establish the fundamental theorem of space curves which states that if curvature and torsion are the given continuous functions of a real variable s, then they determine the space curve uniquely.

1.2 Obectives

After going through this unit, you will be able to:
Define a space curve
Define a regular function
Derive the unit tangent vector
Find the arclength of the of a curve
Find the solution of problems using arclength and tangent & normal.

1.3 Introductory remark about space curves

Definition: A curve is a locus of a point whose position vector r with respect to the fixed origin is a function of single variable U as a parameter.

Definition: A curve in a plane can be given in the parametric form by the equations x = X(u) and y = Y(u) where u ϵ [a,b].

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Example:

The circle $x^2 + y^2 = a^2$ with centre at origin and radius 'a' as a parametric form x = a cosu and y = a sinu.

Definition: *A surface is a locus of a point whose certisian coordinate (x,y,z) are the functions of two independent parameters u,v (say).*

*Thus, x = f(u,v),*y = g(u,v), z = h(u,v)

A curve in a space is given by the equation x =X(u), y = Y(u) and z =Z(u) where uϵ[a,b] can be represented in the parametric form x = a sinθ cosϕ , y = a sinθsinϕ, z = a cosθ

Example:

The sphere $x^2 + y^2 + z^2 = a^2$ with centre at origin and radius â€˜aâ€™ can be represented in the parametric form x = a sinθcosϕ , y = a sinθsinϕ, z = acosθ

Note:

r=xi+yj+zk = x(u)i+y(u)j+z(u)k = R(u)

Definition: A space curve may be expressed as the intersection of two surface f(x,y,z) =0 ......(1) and g(x,y,z) =0 ......(2)

The parametric representation of the space curve is

x =X(u), y =Y(u), z =Z(u) .....(3)

Equation (3) can be transformed into equation (1) and (2) by eliminating u from equation (3).

Problem:1

Find t he equation of the curve whose parametric equations are x=u, y=u ^{2}, y=u^3

Solution:

xy = u.u ^{2} = u ^{3}, xy =z, xz = u. u ^{3}= u ^{4}=(u ^{4})^2, xz = y ^{2}

Definition: Let i be a real interval and m be a positive integer. A real valued function f defined on I is said to be a class of m if

(i) f has m th derivative at every point of i.

(ii) Each derivative is continuous on i.

Definition: A function R is said to be regular if the derivative $\frac{dR}{du}$ = $\dot{R} \neq 0$ on the real interval i.

Definition: A regular vector valued function of the class m is called a path of class m.

Definition: Two paths of R_1 and R $_2$ of same class m on the interval I $_1$ and I $_2$ are said to be equivalent if there exists a strictly increasing function ϕ of class m which map I $_1$ and I $_2$ such that R_1 = R $_2$. ϕ

Theorem:

Derive an expression for the arc length of the curve in space of the form s = s(u) = $\int_{u_o}^{u} |\dot{R}(u)|$du.

Proof:

Let r = R(u) be a path and a,b are two real numbers where a$< u < b$

Take any subdivision Δ of the closed interval a,b.

Now a = $u_0 < u_1 < u_2 <\ldots< u_n$ =b

The corresponding length

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$L\ \triangle = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})| du$

$= \sum_{i=1}^{n} |\int_{u_{i-1}}^{u_i} \dot{R}(u) du|$

$\leq \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} |\dot{R}(u)| du$

$= \int_{u_0}^{u_n} |\dot{R}(u)| du$

$L\ \triangle \leq \int_{u_0}^{u_n} |\dot{R}(u)| du$

Let s =s(u) denote the arc length from a point a to any point then the arc length

from u_0 to u is s(u)-s(u_0)$\leq \int_{u_0}^{u} |\dot{R}(u)| du$ ....(1)

By the definition of arc length,

$$|R(u) - R(u_0)| \leq \int_{u_0}^{u} |\dot{R}(u)| du$$

$|R(u) - R(u_0)| \leq s(u) - s(u_0)$ ....(2)

$\Rightarrow |R(u) - R(u_0)| \leq s(u) - s(u_0)$

$\leq \int_{u_0}^{u} |\dot{R}(u)| du$

$\Rightarrow |\frac{R(u)-R(u_0)}{u-u_0}| \leq \frac{s(u)-s(u_0)}{u-u_0} \leq \frac{1}{u-u_0} \int_{u_0}^{u} |\dot{R}(u)| du$

$\Rightarrow lt_{u\to u_0} \left|\frac{R(u)-R(u_0)}{u-u_0}\right| \leq lt_{u\to u_0} \frac{s(u)-s(u_0)}{u-u_0}$

$$\leq lt_{u\to u_0} \frac{1}{u-u_0} \int_{u_0}^{u} |\dot{R}(u)| du$$

$\Rightarrow |\dot{R}(u_0)| \leq \dot{s}(u_0) \leq |\dot{R}(u_0)|$

$\Rightarrow \dot{s}(u_0) = |\dot{R}(u_0)| \Rightarrow \dot{s}(u) = |\dot{R}(u)|$

$\Rightarrow s = s(u) = \int_{u_0}^{u} |\dot{R}(u)| du$

Problem.2

Expression of arc in cartesian parametric representation is

s =$\int_{a}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} du$

Solution

Let $\vec{r} = (x, y, z) = \vec{R}(u)$

$\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z}) = \dot{R}(u)$

$$|\dot{R}(u)| = |\dot{\vec{r}}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

s =$\int_{a}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} du$

Problem.3

Prove that s = $|\dot{\vec{r}}|$

Solution

We know that s =$\int_{a}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} du$

=$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

s = $|\dot{\vec{r}}|$

Problem.4

Prove that $\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

Solution:

We know that s = $|\dot{\vec{r}}|$

=$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

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4. Prove that $\dot{s}^2 = dx^2 + dy^2 + dz^2$

Solution:

$$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$(\frac{ds}{du}^2) = (\frac{dx}{du}^2) + (\frac{dy}{du}^2) + (\frac{dz}{du}^2)$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

5. Prove that

(i) $\vec{r}$=(a cosu, a sinu, bu), $0 \leq u \leq \infty$

(ii)$\vec{r} = (a.\frac{1-v^2}{1+v^2}, \frac{2av}{1+v^2}, 2btan^{-1}v)$ are equivalent representation for circular helix

Solution:

Given that $\vec{r}$=(a cosu, a sinu, bu) ...(1), $0 \leq u \leq \infty$

$\vec{r} = (a.\frac{1-v^2}{1+v^2}, \frac{2av}{1+v^2}, 2btan^{-1}v)$ ....(2), $0 \leq v \leq \infty$

To prove (1) and (2) are equivalent it is enough to prove that

(i) a $(\frac{1-v^2}{1+v^2})$ = a cosu

(ii) $\frac{2av}{1+v^2}$ = a sinu

(iii) $2btan^{-1}v$ = bu

For (i)

Put v = $tan\frac{u}{2}$

a $(\frac{1-v^2}{1+v^2})$= a $(\frac{1-tan\frac{u}{2}^2}{1+tan\frac{u}{2}^2})$ = a cos 2.$\frac{u}{2}$

⇒ a $(\frac{1-v^2}{1+v^2})$ = acosu

For (ii)

$\frac{2av}{1+v^2} = \frac{2atan\frac{u}{2}}{1+tan\frac{u}{2}^2}$ = a[$\frac{2tan\frac{u}{2}}{1+tan\frac{u}{2}^2}$] = a sin 2.$\frac{u}{2}$

$\frac{2av}{1+v^2}$ = asinu

For(iii)

$2btan^{-1}v$ = $2btan^{-1}tan\frac{u}{2}$ =2b$\frac{u}{2}$

$2btan^{-1}v$ = bu

6. Obtain the equation of the circular helix r =(acosu, asinu, bu), $-\infty < u < \infty$ where a>0 referred to s as parameter and show that the length of one complete turn of the helix is 2Πc where c=$\sqrt{a^2 + b^2}$

(or)

Find the length of the helix $\vec{r}(u) = acosu\vec{\imath} + asinu\vec{\jmath} + bu\vec{k}$, $-\infty < u < \infty$ from (0,0,0) to (a,0,2Πc). Also obtain its equation interms of parameters. Then find the length of the one complete turn of the circular helix $\vec{r} = acosu\vec{\imath} + asinu\vec{\jmath} + bu\vec{k}$, $-\infty < u < \infty$

Solution:

Given r =(acosu, asinu, bu)....(1)

(or) $\vec{r} = acosu\vec{\imath} + asinu\vec{\jmath} + bu\vec{k}$

$$\vec{r} = -asinu\vec{\imath} + acosu\vec{\jmath} + b\vec{k}$$

$$|\vec{r}(u)| = \sqrt{(-asinu)^2 + (acosu)^2 + (b)^2}$$

$$= \sqrt{a^2sin^2u + a^2cos^2u + (b)^2}$$

$$= \sqrt{a^2(sin^2u + cos^2u) + (b)^2}$$

$|\vec{r}(u)| = \sqrt{a^2+b^2}$ ......(2)

w.k.t s = $\int_a^u \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$du

s=$\int_a^u \sqrt{a^2+b^2}$

s =cu

$\Rightarrow u = \frac{s}{c}$ or u =$\frac{s}{\sqrt{a^2+b^2}}$

(1) ⇒ r =$(acos(\frac{s}{c}), asin(\frac{s}{c}), b(\frac{s}{c}))$

$$\vec{r} = acos\frac{s}{\sqrt{a^2+b^2}}\vec{\imath} + asin\frac{s}{\sqrt{a^2+b^2}}\vec{\jmath} + b\frac{s}{\sqrt{a^2+b^2}}\vec{k}$$

which is required equation of the circular helix.

The range of u corresponding to one complete turn of the helix is $u_0 \leq u \leq u_0 \leq u_0 + 2\Pi$

(ie) THe length of limits of one complete turn of the circular helix is o to 2π.

we have $|\vec{\dot{r}}| = \sqrt{a^2+b^2}$

The length of circular helix from o to 2π is

$$s = \int_0^{2\pi} |\vec{\dot{r}}(u)|du$$
$$= \int_0^{2\pi} \sqrt{a^2+b^2}du$$
$$= \sqrt{a^2+b^2}\int_0^{2\pi} du$$
$$= \sqrt{a^2+b^2}[u]_0^{2\pi}$$
$$= \sqrt{a^2+b^2}(2\pi - 0)$$
$$s = 2\pi c$$

Hence proved.

7. Find the length of the curve given as the intersection of the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2}$=1, x = cosh($\frac{z}{a}$) from the point (a,0,0) to (x,y,z)

Solution:

Let the equation of the curve in the parametric form is x =acoshu, y =bsinhu and z =au

w.k.t s = $\int_a^u \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$du

$$\vec{r} = acoshu\vec{\imath} + bsinhu\vec{\jmath} + au\vec{k}$$
$$\vec{\dot{r}} = asinhu\vec{\imath} + bcoshu\vec{\jmath} + a\vec{k}$$
$$|\vec{\dot{r}}(u)| = \sqrt{(asinhu)^2 + (bcoshu)^2 + (a)^2}$$
$$= \sqrt{a^2sinh^2u + b^2cosh^2u + (a)^2}$$
$$= \sqrt{a^2(sinh^2u+1) + (b)^2cosh^2u}$$

$|\vec{\dot{r}}(u)| = cosh^2u\sqrt{(a^2+b^2)}$ ......(2)

s = $\int_a^u \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$du

= $\int_a^u |\vec{\dot{r}}(u)|.du$

= $\int_a^0 coshu\sqrt{(a^2+b^2)}.du$

=$\sqrt{(a^2+b^2)}\int_a^0 coshu.du$

=$\sqrt{(a^2+b^2)}[sinhu]\int_a^0$

=$\sqrt{(a^2+b^2)}sinhu$

=$\sqrt{(a^2+b^2)}(\frac{y}{b})$

s=$\sqrt{(a^2+b^2)}(\frac{y}{b})$

NOTES

Self-Instructional Material

NOTES

Tangent, normal and binomial

Tangent line:

The tangent line to the curve at the point p(u_0) of C is defined as the limiting position of a straight line L through a point p(u_0) and the neighbouring point Q(u) on C as Q tends to P along the curve.

Theorem 1.1

Find the unit tangent vector to the curve.

Proof:

Let γ be any curve represented by $\vec{r} = \vec{r}(u)$ and also γ be a class ≥ 1

Let P and Q be the neighbouring points on the curve represented by the parameter u_0 and u respectively.

Let $\vec{O}P = \vec{r}(u_0)$ and $\vec{O}Q = \vec{r}(u)$,then $\vec{P}Q = \vec{O}Q - \vec{O}P$

$=\vec{r}(u) - \vec{r}(u_0)$, since γ is of class ≥ 1

By taylor's theorem,

$\vec{r}(u) = \vec{r}(u_0) + (u - u_0)\dot{\vec{r}}(u_0) + 0(u - u_0)$ ....(1)

as u tends to u_0.

Now, $lt_{Q\to P} \frac{\bar{r}(u)-\bar{r}(u_0)}{|\bar{r}(u)-\bar{r}(u_0)|} = lt_{Q\to P} \frac{\frac{\bar{r}(u)-\bar{r}(u_0)}{u-u_0}}{\frac{|\bar{r}(u)-\bar{r}(u_0)|}{|u-u_0}|}$

$=\frac{\dot{\vec{r}}(u)}{|\dot{\vec{r}}(u)|}$ [by (1)]

This is called the unit tangent vector to γ at the point P and it is denoted by t.

$$t = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \quad = \frac{\dot{\vec{r}}}{|\dot{\vec{s}}|}$$

$$=\frac{\frac{dr}{du}}{\frac{ds}{du}} = \frac{dr}{ds}$$

$$\vec{t} = \overrightarrow{r'} = \overrightarrow{r'}(u_0)$$

Oscillating plane (or) plane of curvature:

Let γ be a class ≥ 2. Consider the two neighbouring points P and Q on the curve C then the oscillating plane at the point P is the limiting position of the plane which contains the tangent at P and at a point Q has Q tends to P.

Theorem 1.2

Find the equation of the oscillating plane

Proof:

Let γ be a class ≥ 2.

Let P and Q be two neighbouring points on the curve C represented by the parameter O and S respectively.

$$\bar{O}P = \bar{r}(O)$$
$$\bar{O}Q = \bar{r}(s)$$
$$\bar{P}Q = \bar{r}(s) - \bar{r}(o)$$

The unit tangent vector at the point p $=\vec{r}o(o)$

Let $\bar{r}$ be the position vector at the current point P on the plane.

PR = R-$\bar{r}(o)$

$$\bar{t} = \overline{r'}(o)$$

$\bar{P}Q = \bar{r}(s) - \bar{r}(o)$ lie on the same plane.

The equation of the plane is

$[R-\bar{r}(o),\overline{r'}(o),\bar{r}(s)-\bar{r}(o)] = 0$ ......(1)

Also by taylor's expansion

$\bar{r}(s)=\bar{r}(o)+\frac{s}{1!}\overline{r}'(o)+\frac{s^2}{2!}\overline{r}''(o)+o(s^3)$ ....(2)

$\bar{r}(s)-\bar{r}(o)=\frac{s}{1!}\overline{r}'(o)+\frac{s^2}{2!}\overline{r}''(o)+o(s^3)$.....(3)

Sub(3) in (1)

$[R-\bar{r}(o),\overline{r'}(o),\frac{s}{1!}\overline{r}'(o)+\frac{s^2}{2!}\overline{r}''(o)] = 0$

$\Rightarrow [R-\bar{r}(o),\overline{r'}(o),\frac{s}{1!}\overline{r'}(o)]$

$+[R-\ \bar{r}(o),\overline{r'}(o),\frac{s^2}{2!}\overline{r}''(o)] = 0$

$\Rightarrow s[R-\bar{r}(o),\overline{r'}(o),\overline{r'}(o)]$

$+\frac{s^2}{2!}[R-\bar{r}(o),\overline{r}'(o),\overline{r}''(o)] = 0$

$\Rightarrow 0+\frac{s^2}{2!}[R-\bar{r}(o),\overline{r}'(o),\overline{r}''(o)] = 0$

$[R-\bar{r}(o),\overline{r}'(o),\overline{r}''(o)] = 0$

Theorem 1.3

Show that if a curve is given interms of general parameter u then the equation of oscillating plane is $[R-\bar{r}(o),\dot{\bar{r}}(o),\ddot{\bar{r}}(o)] = 0$

Proof:

We know that,

The equation of oscillating plane is

$[R-\bar{r}(o),\overline{r'}(o),\overline{r}''(o)] = 0$ ...(1)

Now, $\overline{r'}=\frac{dr}{ds}=\frac{\frac{dr}{du}}{\frac{ds}{du}}$

$\Rightarrow \overline{r'}=\frac{\dot{\bar{r}}}{\dot{s}}$ ......(2)

$\overrightarrow{r''}=\frac{d}{ds}(\overrightarrow{r'}) = \frac{d}{ds}(\frac{\dot{\vec{r}}}{\dot{s}}) = \frac{d}{du}(\frac{\dot{r}}{\dot{s}}).\frac{du}{ds}$

$=\frac{\dot{s}\ddot{r}-\dot{r}\ddot{s}}{\dot{s}^2}.\frac{1}{\dot{s}} = \frac{\dot{s}\ddot{r}-\dot{r}\ddot{s}}{\dot{s}^3}$ .....(3)

Sub(2), (3) in (1)

(1) $\Rightarrow [R-\bar{r}(o),\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{r}-\dot{r}\ddot{s}}{\dot{s}^3}] = 0$

$\Rightarrow [R-\bar{r}(o),\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{r}}{\dot{s}^3}]-[R-\bar{r}(o),\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{r}\ddot{s}}{\dot{s}^3}] = 0$

$\Rightarrow \frac{\dot{\bar{s}}}{\dot{\bar{s}}.\dot{s}^3}[R-\bar{r}(o),\dot{\bar{r}},\ddot{r}]-\frac{\ddot{s}}{\dot{s}^4}[R-\bar{r}(o),\dot{\bar{r}},\dot{r}] = 0$

$\Rightarrow \frac{1}{\dot{\bar{s}}^3}[R-\bar{r}(o),\dot{\bar{r}},\ddot{r}] = 0$

$\Rightarrow [R-\bar{r}(o),\dot{\bar{r}},\ddot{r}] = 0$

Theorem 1.4

Prove that the cartesian equation of the oscillating planr is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Proof:

We know that,

The equation of oscillating plane is

$[R-\bar{r}(o),\dot{\bar{r}},\ddot{r}] = 0$ ....(1)

$$\vec{R}=X(u)\vec{\imath}+Y(u)\vec{\jmath}+Z(u)\vec{k}$$

$$\vec{r}=x(u)\vec{\imath}+y(u)\vec{\jmath}+z(u)\vec{k}$$

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$$\dot{\vec{r}} = \dot{x}(u)\vec{i} + \dot{y}(u)\vec{j} + \dot{z}(u)\vec{k}$$
$$\ddot{\vec{r}} = \ddot{x}(u)\vec{i} + \ddot{y}(u)\vec{j} + \ddot{z}(u)\vec{k}$$

Sub in (1)

$[(X(u)\vec{i} + Y(u)\vec{j} + Z(u)\vec{k}) - (x(u)\vec{i} + y(u)\vec{j} + z(u)\vec{k}), (\dot{x}(u)\vec{i} + \dot{y}(u)\vec{j} + \dot{z}(u)\vec{k}), (\ddot{x}(u)\vec{i} + \ddot{y}(u)\vec{j} + \ddot{z}(u)\vec{k})]$ =0

$((X - x)(u)\vec{i} + (Y - y)(u)\vec{j} + (Z - z)(u)\vec{k}), (\dot{x}(u)\vec{i} + \dot{y}(u)\vec{j} + \dot{z}(u)\vec{k}), (\ddot{x}(u)\vec{i} + \ddot{y}(u)\vec{j} + \ddot{z}(u)\vec{k})]$ =0

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

3. Find the equation of the oscillating plane at the general point on a cubic curve given by r =(u, u^2, u^3) and show the oscillating planes at any 3 points of the cur e meet at a point lying in the plane determined by these 3 points

Solution:

(i) We know that

The equation of the oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Given that r =(u, u^2, u^3)

(ie) x=u, y=u^2, z=u^3

$$\dot{x} = 1, \dot{y} = 2u, \dot{z} = 3u^2$$
$$\ddot{x} = 0, \ddot{y} = 2, \ddot{z} = 6u$$

$$\begin{vmatrix} X-u & Y-u^2 & Z-u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 0$$

$(X - x)[12u^2 - 6u^3] - (y - u^2)[6u - 0] + (z - u^3)[2 - 0]$=0

$(X - x)[6u^2] - (y - u^2)[6u] + (z - u^3)[2]$=0

$6u^2x - 6uy + 2z - 2u^3$=0

$3u^2x - 3uy + z - u^3$=0

which is the required equation of the oscillating plane.

(ii) Let (u_1, u_2, u_3) be any 3 points on the given curve then the oscillating plane at these 3 points are

$3u_1^2x - 3u_1y + z - u_1^3 = 0$ .....(2)

$3u_2^2x - 3u_2y + z - u_2^3 = 0$ .....(3)

$3u_3^2x - 3u_3y + z - u_3^3 = 0$ .....(4)

Suppose that these 3 planes meet at the point x_0, y_0, z_0. we have,

$$3u^2x_0 - 3uy_0 + z_0 - u^3 = 0$$

$u^3 - 3u^2x_0 + 3uy_0 - z_0 = 0$ ....(5)

Suppose the equation of plane passes through the 3 points (ur, ur^2, ur^3)

λ=(1,2,3) be AX+BY+CZ=1 ....(6)

(u_1, u_2, u_3) be the roots of the equation Au+B$u^2 + Cu^3 - 1$=0 ...(7)

Comparing the eqn(5) and (7)

$$\frac{A}{3Y_0} = \frac{B}{-3X_0} = \frac{C}{1} = \frac{-1}{-Z_0}$$

$\Rightarrow \frac{A}{3Y_0} = \frac{1}{Z_0}$

A=$\frac{3Y_0}{z_0}$, B=$\frac{-3X_0}{z_0}$, C=$\frac{1}{z_0}$

Sub in eqn(6)

$\frac{3Y_0}{z_0} - \frac{-3X_0}{z_0} + \frac{1}{z_0}Z$=1

$$3Y_0X - 3X_0Y + Z = z_0$$

$3Y_0X - 3X_0Y + Z - z_0$=0,

which is the required equation of planes determined by the points (ur, ur^2, ur^3)

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4. Find the equation of the oscillating plane at the point on the helix x =acosu, y=bsinu and z=bu

solution:

Given that $\vec{r}$=(acosu, bsinu, bu)

We know that,

The equation of the oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0 \quad ...(1)$$

x=acosu, y=asinu, z=bu

$$\dot{x} = -asinu, \dot{y} = acosu, \dot{z} = b$$

$$\ddot{x} = -acosu, \ddot{y} = -asinu, \ddot{z} = 0$$

$$\begin{vmatrix} X-acosu & Y-asinu & Z-bu \\ -asinu & acosu & b \\ -acosu & -asinu & 0 \end{vmatrix} = 0$$

$\Rightarrow (X - acosu)[0 + absinu] - (Y - asinu)[0 + abcosu] + (z - bu)[a^2sin^2u + a^2cos^2u$=0

$\Rightarrow xabsinu - a^2bsinucosu - yabcosu + a^2bsinucosu + za^2 - bua^2$=0

$\Rightarrow xabsinu - yabcosu + za^2 - bua^2$=0

$\Rightarrow xbsinu - ybcosu + za - bua$=0,

which is the required equation of oscillating plane.

5. Show that when the curve is analytic obtain a definite oscillating plane at the point of inflection 'p' unless the curve is the straight line

Solution:

We know that, $\bar{t} = \overline{r'}$and $\bar{t}.\bar{t} = \overline{r'}.\overline{r'}$

$\bar{t}.\bar{t} = 1$ ....(1)

Diff ėqn(̇1) w.r to s

$\overline{r'}.\overline{r''} + \overline{r'}.\overline{r''} = 0$ $2\overline{r'}.\overline{r''}$=0 ....(2)

$\overline{r'}.\overline{r''}$=0

Diff.ėqn(̇2) w.r to s

$\overline{r'}.\overline{r'''} + \overline{r''}.\overline{r''} = 0$ $\overline{r'}.\overline{r'''} + 0$=0

$\overline{r'}.\overline{r'''}$=0....(3), since r"=0

Case 1

If $\overrightarrow{r'''}$ ≠0 then $\overline{r'}$ and $\overrightarrow{r'''}$ are linearly independent.

The equation of the oscillating plane is $[\bar{R} - \bar{r}, \overline{r'}, \overline{r'''}]$ =0

Case 2

If $\overrightarrow{r'''}$ =0

Diff. (̇3) w.r to s

$\overrightarrow{r'}.\vec{r}^{iv} + \overrightarrow{r'''}.\overrightarrow{r''}$=0

$\overrightarrow{r'}.\vec{r}^{iv} + 0$=0,

$\overrightarrow{r'}.\vec{r}^{iv}$=0 ....(4), since $\overrightarrow{r'''} = 0$

NOTES

In general $\overrightarrow{r'}.\vec{r}^k$=0 ....(4)

If $\vec{r}^k$ ≠0 for k≥ 2

Then the equation of the oscillating plane becomes $[\bar{R}-\bar{r},\overline{r'},\bar{r}^k]$ =0

If $\vec{r}^k$ =0 for k≥ 2, then $\overrightarrow{r''} = 0$

(ie) $\overrightarrow{r'} = constant,$ t = constant

The curve is a straight line. since it is analytic.

Hence the point of inflection even a class infinity of the curve need not posses an oscillating plane.

Normal plane:

The plane through the point P which is normal to the tangent at P is called the normal plane on the curve.

The equation of the normal plane is

$$(\bar{R}-\bar{r}).\overrightarrow{r'} = 0$$

(ie) $(\bar{R}-\bar{r}).\bar{t}$=0

Principle normal:

The line of intersection of the normal plane and oscillating plane is called a principle normal at a point p.

Binormal:

The normal which is perpendicular to the oscillating plane at a point p is called binormal at p.

Note:

The unit vector along the principle normal and binormal are denoted by $\vec{n}$ and $\vec{b}$.

The unit vectors are $\vec{t} = \vec{n}\times\vec{b}$, $\vec{b} = \vec{t}\times\vec{n}$, $\vec{n} = \vec{b}\times\vec{t}$

$\vec{n}.\vec{b}$=0, $\vec{t}.\vec{n}$=0,$\vec{t}.\vec{b}$=0

(i) The oscillating plane containing $\bar{t}$ and $\bar{n}$ and its equation is $(\bar{R}-\bar{r}).\bar{b}$=0

(ii) The normal plane containing $\bar{n}$ and $\bar{b}$ and its equation is $(\bar{R}-\bar{r}).\bar{t}$=0

(iii) The principle plane containing $\bar{n}$ and $\bar{t}$ and its equation is $(\bar{R}-\bar{r}).\bar{n}$=0

1.4 Check your progress

1.Define arc length

2. Define space curve

3. Define regular vector valued function

4. Define a function of class m

5. Define tangent, normal and binormal

1.5 Summary

A curve is a locus of a point whose position vector r with respect to the fixed origin is a function of single variable U as a parameter.

A curve in a plane can be given in the parametric form by the equations x = X(u) and y = Y(u) where u ϵ [a,b].

A space curve may be expressed as the intersection of two surface f(x,y,z) =0 and g(x,y,z) =0 ......(2)

The arc length of the curve in space of the form s = s(u) = $\int_{u_o}^{u} |\dot{R}(u)|$du.

The tangent line to the curve at the point p(u_0) of C is defined as the limiting position of a straight line L through a point p(u_0) and the neighbouring point Q(u) on C as Q tends to P along the curve.

The cartesian equation of the oscillating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}=0$$

NOTES

The equation of the normal plane is $(\bar{R}-\bar{r}).\overrightarrow{r'}=0$

The unit vectors are $\vec{t}=\vec{n}\times\vec{b}, \vec{b}=\vec{t}\times\vec{n}, \vec{n}=\vec{b}\times\vec{t}$

$\vec{n}.\vec{b}$=0, $\vec{t}.\vec{n}$=0,$\vec{t}.\vec{b}$=0

(i) The oscillating plane containing $\bar{t}$ and $\bar{n}$ and its equation is $(\bar{R}-\bar{r}).\bar{b}$=0

(ii) The normal plane containing $\bar{n}$ and $\bar{b}$ and its equation is $(\bar{R}-\bar{r}).\bar{t}$=0

(iii) The principle plane containing $\bar{n}$ and $\bar{t}$ and its equation is $(\bar{R}-\bar{r}).\bar{n}$=0

1.6 Keywords

Curve: A curve is a locus of a point whose position vector r with respect to the fixed origin is a function of single variable U as a parameter

Arclength: The arc length of the curve in space of the form s = s(u) = $\int_{u_o}^{u}|\dot{R}(u)|$du

Oscillating plane (or) plane of curvature: Let γ be a class ≥ 2. Consider the two neighbouring points P and Q on the curve C then the oscillating plane at the point P is the limiting position of the plane which contains the tangent at P and at a point Q has Q tends to P.

Regular function: A function R is said to be regular if the derivative $\frac{dR}{du}=\dot{R}\neq 0$ on the real interval i.

1.7 Self Assessment Questions and Exercises

1. Determine the function f(u) so that the curve given by r=(a cosu, a sinu, f(u)) shall be plane.
2. Find the equation of the osculating plane at a point on the helix r= (a cos u, a sin u, au tan α)
3. Find the equation of the osculating plane at a general point on the cubic curve r=(u, u^2, u^3) and show that the osculating planes at any three points of the curve meet at a point lying in the plane determined by these three points.

4. Find the coordinates of the centre of spherical curvature of the curve given by
r=(acosu, asinu, acos2u).

5. Prove that the curve given by x=a$\sin^2$u, y=asinu cosu, z=acosu lies on a sphere.

6. Show that the principal normal to a curve is normal to the locus of centres of curvature at those points where the curvature is stationary.

7. Determine the form of the function ϕ(u) such that the principal normal of the curve r=(a cos u, a sin u, ϕ(u)) are parallel to the XOY plane.

8. Show that the principal normal at two consecutive points of a curve do not intersect unless τ=0.

NOTES

9. If there is a one-one correspondence between the points of two curves and tangents at the corresponding points are parallel, show that the principal normals are parallel and so also their binormals. Also prove that $\frac{\kappa_1}{\kappa} = \frac{ds}{ds_1} = \frac{\tau_1}{\tau}$

10. A pair of curves Υ, Υ_1 which have the same principal normals are called Bertrand curves. Prove that the tangents to Υ and Υ_1 are inclined at a constant angle, and show that, for each curve there is a linear relation with constant coefficients between the curvature and torsion.

1.8Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).

2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010).

UNIT –II CURVATURE AND TORSION OF A CURVE

Structure

2.1 Introduction

At each point of the curve, we have defined an orthogonal triad t, n, b forming a right handed system and also we have noted that at each point this moving triad determines three fundamental planes which are mutually perpenticular. Hence we can study their variations from point to point with respect to the arcual length as parameter. This leads to the notion of curvature and torsion of the space curve as defined below. Using the concept of curvature and torsion, it is easily to derive Serret-Frenet formulae.

2.2 Objectives

After going through this unit, you will be able to :

- Define curvature and torsion of a curve
- Solve the problems using curvature and torsion
- Derive Serret-Frenet formulae
- Derive the necessary and sufficient conditions for a curve to be a straight line or a curve.

2.3 Curvature and torsion of a curve

Curvature:

The arc rate at which the tangent changed direction as p moves along the curve is called curvature of the curve and its denoted by κ

By definition $|\kappa| = |t'|$

Torsion:

As p moves along the curve the arc rate at which the oscillating plane turns about the tangent is called the torsion of a curve and its denoted by τ

6. Derivation of serret-Frenet formula:

(i) Prove that $\frac{d\bar{t}}{ds} = \kappa\bar{n}$

(ii) $\frac{d\bar{n}}{ds} = \tau\bar{b} - \kappa\bar{t}$

(iii)$\frac{d\bar{b}}{ds} = -\tau\bar{n}$

Solution:

(i) To prove :$\frac{d\bar{t}}{ds} = \kappa\bar{n}$

We know that,

$$\overline{r'} = \frac{dr}{ds} = \frac{dr}{du}.\frac{du}{ds} = \dot{\vec{r}}.u'$$

$$\overrightarrow{r''} = \frac{d}{ds}(\overrightarrow{r'}) = \frac{d}{ds}(\dot{\vec{r}}.u')$$

$$= \dot{\bar{r}}.\frac{du'}{ds} + \frac{d\dot{\bar{r}}}{ds}.u' = \dot{\bar{r}}.u'' + \frac{d\dot{\bar{r}}}{du}.\frac{du}{ds}.u'$$

$$= \dot{\bar{r}}.u'' + \ddot{\bar{r}}.(u')^2$$

Therefore, $\overline{r''}$ lies in the oscillating plane.

We know that,

$\bar{t}$ is a unit tangent vector at the point p and

$\bar{t}.\bar{t} = 1$.....(1)

Diff(1) w.r to s,

$$\bar{t}.\frac{d\bar{t}}{ds} + \frac{d\bar{t}}{ds}.\bar{t} = 0$$

$$2\bar{t}.\frac{d\bar{t}}{ds} = 0$$

$$\bar{t}.\frac{d\bar{t}}{ds} = 0$$

$$\bar{t}.\overrightarrow{r''} = 0 \text{ ....(2)}$$

(ie) $\overrightarrow{r''}$ is perpendicular to $\bar{t}$.

From eqn (1) and (2), we have $\overrightarrow{r''}$ coincide with $\bar{n}$

(ie) $\frac{d\bar{t}}{ds} = \kappa\bar{n}$

Hence (i) is proved.

(ii) we know that $\bar{n} = \bar{b} \times \bar{t}$

Diff w.r to s,

$$\frac{d\bar{n}}{ds} = (\frac{d\bar{b}}{ds} \times \bar{t}) + (\bar{b} \times \frac{d\bar{t}}{ds})$$

$$= (-\tau\bar{n} \times \bar{t}) + (\bar{b} \times \kappa\bar{n})$$

$$= -\tau(\bar{n} \times \bar{t}) + \kappa(\bar{b} \times \bar{n})$$

$$= -\tau(-\bar{b}) + \kappa(-\bar{t}) = (\tau\bar{b}) - (\kappa\bar{t})$$

$$\frac{d\bar{n}}{ds} = \tau\bar{b} - \kappa\bar{t}$$

(iii) we know that, $\bar{t}$ is perpendicular to $\bar{b}$

(ie) $\bar{t}.\bar{b} = 0$ ....(3)

Diff w.r.to s,

$$(\frac{d\bar{t}}{ds}.\bar{b}) + (\bar{t}.\frac{d\bar{b}}{ds}) = 0$$

$$\Rightarrow \kappa\bar{n}.\bar{b} + \bar{t}.\frac{d\bar{b}}{ds} = 0$$

$$\Rightarrow \kappa(\bar{n}.\bar{b}) + \bar{t}.\frac{d\bar{b}}{ds} = 0$$

$$\Rightarrow \kappa.0 + \bar{t}.\frac{d\bar{b}}{ds} = 0$$

$$\Rightarrow \bar{t}.\frac{d\bar{b}}{ds} = 0$$

Therefore, $\bar{t}$ is perpendicular to $\frac{d\bar{b}}{ds}$ ....(4)

We know that,

$\bar{b}$ is the unit binormal

(ie) $\bar{b}.\bar{b} = 1$

Diff w.r to s,

$$\bar{b}.\frac{d\bar{b}}{ds} + \frac{d\bar{b}}{ds}.\bar{b} = 0,$$

$2\bar{b}.\frac{d\bar{b}}{ds}$=0
$\bar{b}.\frac{d\bar{b}}{ds}$=0
$\bar{b}$ is perpendicular to $\frac{d\bar{b}}{ds}$ ....(5)
From (4) and (5) we have
$\frac{d\bar{b}}{ds}$ coincide with $\bar{n}$
(ie) $|\frac{d\bar{b}}{ds}| = \tau\bar{n}$
Therefore, $\frac{d\bar{b}}{ds} = -\tau\bar{n}$

Theorem 1.5

Show that the necessary and sufficient that a curve to be a straight line is that κ=0 at all points.

Proof:

Necessary part:
The equation of the straight line is $\bar{r} = \bar{a}s + \bar{b}$....(1), where $\bar{a}$ and $\bar{b}$ are constant vector.
Diff(1) w.r to s,
$\frac{d\bar{r}}{ds} = \bar{a} \Rightarrow \overline{r'} = \bar{a} \Rightarrow \bar{t} = \bar{a}$ ...(2)
Diff(2) w.r to s
$\frac{d\bar{t}}{ds}$=0 $\Rightarrow t' = 0$
But we know that
$\frac{d\bar{t}}{ds}$=$\kappa\bar{n}$ $\Rightarrow \kappa\bar{n} = 0$
$\kappa = 0$ at all points.
Sufficient part:
If $\kappa = 0$ then $\frac{d\bar{t}}{ds} = 0$
(ie) t' =0
(ie)$\overline{r''}$=0 ....(3)
Integrating (3) w.r to s
$\overline{r'}$=a
$\bar{r} = as + b$ ..(4) where a and b are constant vectoer.
Here eqn(4) represents straight line.

Theorem 1.6

Prove that a necessary and sufficient condition that a curve γ be a plane curve is that τ=0 at all points.

Proof:

Necessary part:
Assume that: A curve is a plane curve.
To prove: τ=0 at all points.
Since the curve is a plane curve the tangent and normalat all points are also lie on the plane.
The plane is an oscillating plane for all points.
⇒ The binormal vector is same at all points.
The binormal vector is constant.
$\Rightarrow \bar{b} = \bar{c}$ where $\bar{c}$ is a constant vector.
$\Rightarrow \frac{d\bar{b}}{ds}$=0
$\Rightarrow |\frac{db}{ds}|$=0

τ =0 at all points.
Sufficient part:
Assume that: τ=0 at all points.
To prove: The curve γ is a plane curve.
Now τ=0 ⇒ $|\frac{db}{ds}|$=0
⇒ $|b'|$=0 ⇒ $b' = 0$
Integrating w. r to s
$\vec{b} = \vec{c}$ where c is a constant vector.
⇒ The binormal vector is same at all points.
The plane is an oscillating plane.
The tangent plane and normal at all points are also lie on the plane.
The curve is a plane curve.

Theorem 1.7

Show that the necessary and sufficient condition for the curve to be the plane curve is $[\overrightarrow{r'}, \overrightarrow{r''}, \overrightarrow{r'''}]=0$

Proof:

We have $\overrightarrow{r'} = \frac{d\bar{r}}{ds}$

⇒ $\overline{r'}$=t ⇒ $\overline{r''} = t'$

But $\frac{dt}{ds} = \kappa\bar{n}$

(ie) $\overline{r''} = \kappa\bar{n}$

$$\overline{r'''} = \frac{d}{ds}(\overline{r''})$$
$$=\frac{d}{ds}(\kappa\bar{n})$$
$$=\kappa.\frac{d\bar{n}ds}{+}\frac{d\bar{k}}{ds}.\bar{n}$$
$$=\kappa(\tau\bar{b} - \kappa\bar{t}) + \kappa'\bar{n}$$
$$=\kappa\tau\bar{b} - \kappa^2\bar{t} + \kappa'\bar{n}$$
$$=-\kappa^2\bar{t} + \kappa'\bar{n} + \kappa\tau\bar{b}$$

$$[\overrightarrow{r'}, \overrightarrow{r''}, \overrightarrow{r''}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \end{vmatrix} =0$$
$$=1(\kappa^2\tau) - 0 + 0$$
$$=\kappa^2\tau \text{ ....(1)}$$

But given $[\overline{r'}, \overline{r''}, \overline{r'''}]$ =0
(1) becomes $\kappa^2\tau$=0
Either κ =0 (or) τ=0
Now let $\tau \neq 0$ at some points of the curve there is the neighbouring of this points $\tau \neq 0$
$\kappa = 0$ is this neighbouring and hence the curve is a straight line.
τ=0 on this line which is ⇒⇐ to our assumption.
Thus $\tau = 0$ at all points and the curve is a plane curve.
Sufficient part:
Assume that: $\tau = 0$
⇒ The curve is a plane curve.
From (1) $[\overline{r'}, \overline{r''}, \overline{r'''}]$=0
This is a sufficient condition for that a plane.

7. Show that $[\dot{r},\ddot{r},\dddot{r}]$=0 is necessary and sufficient condition that a curve to ba a plane

Solution:

We have $\overline{r'} = \frac{dr}{ds}$

$\overline{r'} = \frac{dr}{du}.\frac{du}{ds} = \dot{r}.u'$

$\overline{r'}' = \frac{d\overline{r'}}{ds}$=$\frac{d}{ds}(\dot{r}u')$ =$\frac{d\dot{r}}{ds}.u' + \dot{r}.\frac{du'}{ds}$

=$\frac{d\dot{r}}{du}.\frac{du}{ds}.u' + \dot{r}u''$ = $\ddot{r}.\frac{du}{ds}.u' + \dot{r}u''$

=$\ddot{r}.u'^2 + \dot{r}.u''$

$\overline{r'}'' = \frac{\overline{r'}'}{ds}$=$\frac{d}{ds}[\ddot{r}u'^2 + \dot{r}u'']$

=$\frac{d\ddot{r}}{ds}.u'^2 + \ddot{r}\frac{du'^2}{ds} + \frac{d\dot{r}}{ds}u'' + \dot{r}\frac{du''}{ds}$

=$\frac{d\ddot{r}}{du}.\frac{du}{ds}.u'^2 + \ddot{r}.2u'u'' + \frac{d\dot{r}}{du}\frac{du}{ds}u'' + \dot{r}u'''$

=$\dddot{r}u'^3 + \ddot{r}.2u'u'' + \ddot{r}u'u'' + \dot{r}u'''$

=$\dddot{r}u'^3 + \ddot{r}.3u'u'' + \dot{r}u'''$

Now,

$[\overline{r'},\overline{r'}',\overline{r'}''] = [\dot{r}u',\ddot{r}u'^2 + \dot{r}u'',\dddot{r}u'^3 + \ddot{r}.3u'u'' + \dot{r}u''']$

=$[\dot{r}u',\ddot{r}u'^2 + \dot{r}u'',\dddot{r}u'^3] + [\dot{r}u',\ddot{r}u'^2 + \dot{r}u'',\ddot{r}.3u'u''] + [\dot{r}u',\ddot{r}u'^2 + \dot{r}u'',\dot{r}u''']$

=$[\dot{r}u',\ddot{r}u'^2,\dddot{r}u'^3] + [\dot{r}u',\dot{r}u'',\dddot{r}u'^3] + [\dot{r}u',\ddot{r}u'^2,\ddot{r}.3u'u''] + [\dot{r}u',\dot{r}u'',\ddot{r}.3u'u''] + [\dot{r}u',\ddot{r}u'^2,\dot{r}u'''] + [\dot{r}u',\dot{r}u'',\dot{r}u''']$

=$[\dot{r}u',\ddot{r}u'^2,\dddot{r}u'^3]$+0+0+0+0+0

$[\overline{r'},\overline{r'}',\overline{r'}''] = [u']^6[\dot{r},\ddot{r},\dddot{r}]$

$[\dot{r},\ddot{r},\dddot{r}] = [u']^{-6}[\overline{r'},\overline{r'}',\overline{r'}'']$

$[\dot{r},\ddot{r},\dddot{r}] = [u']^{-6}(\kappa^2\tau)$

Suppose $[\dot{r},\ddot{r},\dddot{r}] = 0$

(ie) $[u']^{-6}(\kappa^2\tau)$=0 .....(1)

$\Rightarrow \kappa^2\tau$=0

$\Rightarrow \kappa$=0 (or)τ=0

Suppose $\tau \neq 0$ (ie) $\kappa = 0$

The curve is a straight line.

The binormal vector is same at all points.

$\bar{b}$ is constant

$|\frac{d\bar{b}}{ds}|$ =0

$\Rightarrow \tau = 0$, which is a ⇒⇐

$\tau = 0$ at all points.

Sufficient part:

Assume that: $\tau = 0$

The curve is a plane curve.

From(1), we have $[\dot{r},\ddot{r},\dddot{r}] = 0$

This is the sufficient condition for the plane curve.

8. Prove that $k = |\overline{r'} \times \overline{r'}'|$=$\frac{|\dot{r}\times\ddot{r}|}{|\dot{r}^3|}$

Solution:

We know that $\overline{r'} = t$

$\overline{r'}' = \frac{dr'}{ds} = \frac{dt}{ds} = \kappa\bar{n}$

$\overline{r'} \times \overline{r''} = \bar{t} \times \kappa\bar{n} = \kappa(\bar{t} \times \bar{n}) = \kappa\bar{b}$

$|\overline{r'} \times \overline{r''}| = |\kappa|$ ...(1)

Now, $\overline{r'} = \frac{dr}{ds} = \frac{dr}{du}.\frac{du}{ds} = \frac{\dot{\bar{r}}}{\dot{s}}$

$\overline{r'}' = \frac{d\bar{r}}{ds} = \frac{d}{ds}(\frac{\dot{\bar{r}}}{\dot{s}}) = \frac{d}{du}(\frac{\dot{\bar{r}}}{\dot{s}})\frac{du}{ds} = \frac{\frac{d}{du}(\frac{\dot{\bar{r}}}{\dot{s}})}{\dot{s}} = \frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^2}.\frac{1}{\dot{s}}$

$$\overline{r''} = \frac{\dot{s}\ddot{\bar{r}} - \dot{\bar{r}}\dot{s}}{\dot{s}^3}$$

$\overline{r'} \times \overline{r''} = \frac{\dot{\bar{r}}}{\dot{s}} \times \frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s}s}{\dot{s}^3} = \frac{\dot{\bar{r}}}{\dot{s}} \times \frac{\dot{s}\ddot{\bar{r}}}{\dot{s}^3} - \frac{\dot{\bar{r}}}{\dot{s}} \times \frac{\dot{\bar{r}}\ddot{s}}{\dot{s}^3}$

$= \frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{\dot{s}^3} - \frac{\ddot{s}}{\dot{s}^4}(\dot{\bar{r}} \times \dot{\bar{r}})$

$|\overline{r'} \times \overline{r''}| = \frac{|\dot{\bar{r}}\times\dot{\bar{r}}|}{|\dot{\bar{r}}^3|}$ ...(2)

From (1) and (2)

$k = |\overline{r'} \times \overline{r''}| = \frac{|\dot{\bar{r}}\times\dot{\bar{r}}|}{|\dot{\bar{r}}^3|}$

9. Prove that $\boldsymbol{\tau} = \frac{[\overline{r'},\overline{r''},\overline{r'''}]}{|\overline{r'}\times\overline{r''}|^2} = \frac{[\vec{\dot{r}},\vec{\ddot{r}},\vec{\dddot{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}$

Solution:

We know that, $\overline{r'} = t$

$$\overline{r''} = \frac{d\overline{r'}}{ds} = \frac{d}{ds}(t)$$

$$\overline{r'''} = \frac{d\overline{r''}}{ds} = \frac{d}{ds}(\kappa\bar{n})$$

$$= \kappa\frac{d\bar{n}}{ds} + \bar{n}\frac{d\kappa}{ds}$$

$$= \kappa(\tau\bar{b} - \kappa\bar{t}) + n\kappa'$$

$$\overline{r'''} = -\kappa^2\bar{t} + \kappa'\bar{n} + \kappa\tau\bar{b}$$

$$\left[\overrightarrow{r'}, \overrightarrow{r'}', \overrightarrow{r'}''\right] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ -\kappa^2 & \kappa' & \kappa\tau \end{vmatrix} = \kappa^2\tau$$

$\overline{r'} \times \overline{r''} = \bar{t} \times \kappa\bar{n} = \kappa(\bar{t} \times \bar{n})$

$$\overline{r'} \times \overline{r''} = \kappa\bar{b}$$

$\left|\overline{r'} \times \overline{r'}'\right| = \left|\kappa\bar{b}\right| = |\kappa| = \kappa$

$$\Rightarrow |\overline{r'} \times \overline{r''}|^2 = \kappa^2$$

$$\frac{[\overline{r'},\overline{r''},\overline{r'''}]}{|\overline{r'}\times\overline{r''}|^2} = \frac{\kappa^2\tau}{\kappa^2} = \tau \text{ .....(1)}$$

Also, $\overline{r'} = \frac{dr}{ds} = \frac{dr}{du}.\frac{du}{ds} = \frac{\dot{\bar{r}}}{\dot{s}}$

$\overline{r''} = \frac{d\bar{r}}{ds} = \frac{d}{ds}(\frac{\dot{\bar{r}}}{\dot{s}}) = \frac{d}{du}(\frac{\dot{\bar{r}}}{\dot{s}})\frac{du}{ds} = \frac{\frac{d}{du}(\frac{\dot{\bar{r}}}{\dot{s}})}{\dot{s}} = \frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^2}.\frac{1}{\dot{s}}$

$$\overline{r''} = \frac{\dot{s}\ddot{\bar{r}} - \dot{\bar{r}}\dot{s}}{\dot{s}^3}$$

$\overline{r'''} = \frac{d}{ds}(\overline{r''}) = \frac{d}{ds}(\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3})$

$= \frac{d}{du}(\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3}).\frac{du}{ds} = \frac{\frac{d}{du}(\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3})}{\dot{s}}$

$= \frac{\dot{s}^3[(\dot{s}\dddot{\bar{r}}+\ddot{\bar{r}}\ddot{s})-(\ddot{s}\ddot{\bar{r}}+\dddot{s}\dot{\bar{r}})]-(\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s})3\dot{s}^2\ddot{s}}{\dot{s}^6\dot{s}}$

$= \frac{\dot{s}^3(\dot{s}\dddot{\bar{r}}+\ddot{\bar{r}}\ddot{s}-\ddot{s}\ddot{\bar{r}}-\dddot{s}\dot{\bar{r}}-3\dot{s}^3\ddot{\bar{r}}\ddot{s}+3\dot{\bar{r}}\ddot{s}^2\dot{s}^2}{\dot{s}^7\dot{s}}$

$=\frac{\dot{s}^4\dddot{\bar{r}}+\dot{s}^3\ddot{\bar{r}}\ddot{s}-\dot{s}^3\ddot{s}\ddot{\bar{r}}-\dddot{s}\dot{\bar{r}}-3\dot{s}^3\ddot{\bar{r}}\ddot{s}+3\dot{\bar{r}}\ddot{s}^2\dot{s}^2}{\dot{s}^7\dot{s}}$

$\left[\overrightarrow{r'},\overrightarrow{r'}',\overrightarrow{r'}''\right] = \left[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3},\frac{\dot{s}^4\dddot{\bar{r}}+\dot{s}^3\ddot{\bar{r}}\ddot{s}-\dot{s}^3\ddot{s}\ddot{\bar{r}}-\dddot{s}\dot{\bar{r}}-3\dot{s}^3\ddot{\bar{r}}\ddot{s}+3\dot{\bar{r}}\ddot{s}^2\dot{s}^2}{\dot{s}^7\dot{s}}\right]$

$=[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3},\frac{\dot{s}^4\dddot{\bar{r}}}{\dot{s}^7}]$ $-[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3},\frac{\dot{s}^3\ddot{\bar{r}}\ddot{s}}{\dot{s}^7}]$ $-[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3},\frac{3\dot{s}^3\ddot{\bar{r}}\ddot{s}}{\dot{s}^7}]$ $-[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\dot{s}}{\dot{s}^3},\frac{3\dot{\bar{r}}\ddot{s}^2\dot{s}^2}{\dot{s}^7}]$

$=[\frac{\dot{\bar{r}}}{\dot{s}},\frac{\dot{s}\ddot{\bar{r}}}{\dot{s}^3},\frac{\dot{s}^4\dddot{\bar{r}}}{\dot{s}^7}]$-0-0-0-0-0-0-0

$=\frac{\dot{s}^5}{\dot{s}^7}[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]$

$=\frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{\dot{s}^6}$ ...(1)

$\overline{r'}\times\overline{r''} = \frac{\dot{\bar{r}}}{\dot{s}}\times\frac{\dot{s}\ddot{\bar{r}}-\dot{\bar{r}}\ddot{s}s}{\dot{s}^3}$ $=\frac{\dot{\bar{r}}}{\dot{s}}\times\frac{\dot{s}\ddot{\bar{r}}}{\dot{s}^3}-\frac{\dot{\bar{r}}}{\dot{s}}\times\frac{\dot{\bar{r}}\ddot{s}}{\dot{s}^3}$

$=\frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{\dot{s}^3}-\frac{\ddot{s}}{\dot{s}^4}(\dot{\bar{r}}\times\dot{\bar{r}})$

$$|\overline{r'}\times\overline{r''}| = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{\dot{s}^3}$$

$|\overline{r'}\times\overline{r''}|^2 = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{\dot{s}^6}$ ...(2)

From(1)

$$\Rightarrow [\overline{r'},\overline{r''},\overline{r'''}] = \frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{\dot{s}^6}$$

$$\Rightarrow [\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}] = \dot{s}^6[\overline{r'},\overline{r''},\overline{r'''}]$$

From (2)

$$|\overline{r'}\times\overline{r''}|^2 = \frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}{\dot{s}^6}$$

$$|\dot{\bar{r}}\times\ddot{\bar{r}}|^2 = \dot{s}^6|\overline{r'}\times\overline{r''}|^2$$

$$\Rightarrow \frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2} = \frac{\dot{s}^6[\overline{r'},\overline{r''},\overline{r'''}]}{\dot{s}^6|\overline{r'}\times\overline{r''}|^2}$$

$\Rightarrow \frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2} = \frac{[\overline{r\prime},\overline{r\prime\prime},\overline{r\prime\prime\prime}]}{|\overline{r\prime}\times\overline{r\prime\prime}|^2}$=0

10. Determine the function f(u) so that the curve given by $\vec{r}$=(acosu,asinu,f(u)) should be a plane.

Solution:

Given that $\vec{r}$=(acosu,asinu,f(u)) and also a curve is a plane curve.

(ie) τ=0 $\Rightarrow \frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2} = 0$

$\Rightarrow [\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]$=0

$\bar{r}$=(acosu, asinu, f(u))

$$\dot{\bar{r}} = (-asinu, acosu, f'(u))$$

$\ddot{\bar{r}}$ =(-acosu, -asinu, f''(u))

$\dddot{\bar{r}}$ =(asinu, -acosu, f'''(u))

$\Rightarrow [\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]$=0

$$\begin{vmatrix} -asinu & acosu & f'(u) \\ -acosu & -asinu & f''(u) \\ asinu & -acosu & f'''(u) \end{vmatrix} = 0$$

NOTES

$$\Rightarrow \begin{vmatrix} 0 & 0 & f'(u)+f'''(u) \\ -acosu & -asinu & f''(u) \\ asinu & -acosu & f'''(u) \end{vmatrix} = 0 \ (R_1 \to R^1 + R^3)$$

$\Rightarrow (f'(u)+f'''(u))(a^2cos^2u + a^2sin^2u)$=0

$\Rightarrow (f'(u)+f'''(u))(a^2)$=0

$\Rightarrow (f'(u)+f'''(u))$=0 ....(1)

Integrating w.r to u,

f(u)+f''(u)=c where c is a constant

$\Rightarrow f(u) + \frac{d^2}{du^2} f(u)$=c

$\Rightarrow [1 + \frac{d^2}{du^2}] f(u)$=c

$\Rightarrow (D^2+1)f(u) = c$ ....(2), where D = $\frac{d}{du}$

This is the second order D.E

The solution is f(u)=C.F+P.I

To find C.F:

The A.E of (2) is m^2+1=0

$\Rightarrow$ m=$\pm$i

C.F =$e^{0u}(Acosu + Bsinu)$

P.I =$\frac{1}{f(D)} x$ =$\frac{1}{D^2+1} c$ =$\frac{e^{0u}}{D^2+1} c$ = c

The solution is f(u) = C.F+P.I

(ie) f(u) = Acosu+Bsinu+c

11. Calculate the curvature and torsion in the cubic curve is given by $(\boldsymbol{u}, \boldsymbol{u^2}, \boldsymbol{u^3})$

Solution:

We know that

κ=$\frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}{|\dot{\bar{r}}|^3}$ (Curvature), τ=$\frac{[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$ (Torsion)

Given, $\bar{r}$=(u, u^2, u^3),

$$\dot{\bar{r}} = (1, 2u, 3u^2)$$

$$\ddot{\bar{r}} = (0, 2, 6u)$$

$$\dddot{\bar{r}} = (0, 0, 6)$$

$$|\bar{r} \times \ddot{\bar{r}}| = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$

$$= \vec{t}(12u^2 - 6u^2) - \vec{n}(6u) + \vec{b}(2)$$

$$= 6u^2\vec{t} - 6u\vec{n} + 2\vec{b}$$

$$|\bar{r} \times \ddot{\bar{r}}| = \sqrt{36u^4 + 36u^2 + 4}$$

$$= \sqrt{4(9u^4 + 9u^2 + 1)}$$

$$= 2\sqrt{9u^4 + 9u^2 + 1}$$

Also, $|\dot{\bar{r}}| = \sqrt{9u^4 + 9u^2 + 1}$

$$= (1 + 9u^2 + 9u^4)^{\frac{1}{2}}$$

$$|\dot{\bar{r}}|^3 = (1 + 9u^2 + 9u^4)^{\frac{3}{2}}$$

$$\kappa = \frac{2\sqrt{9u^4 + 9u^2 + 1}}{(1 + 9u^2 + 9u^4)^{\frac{3}{2}}}$$

$[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]=\begin{bmatrix}1 & 2u & 3u^2\\0 & 2 & 6u\\0 & 0 & 6\end{bmatrix}$

$=1(12)-2u(0)+3u^2(0)$

$=12$

$$|\dot{\bar{r}}\times\ddot{\bar{r}}|=2\sqrt{9u^4+9u^2+1}$$
$$|\dot{\bar{r}}\times\ddot{\bar{r}}|^2=4(9u^4+9u^2+1)$$
$$\tau=\frac{12}{4(1+9u^2+9u^4)}$$
$$\tau=\frac{3}{1+9u^2+9u^4}$$

12. Find the coordinates of a point interms of s.

Solution:

Let P be the point on the given curve.

Take 'O' as origin and the axes OX, OY, OZ be along $\vec{t},\vec{n},\vec{b}$ respectively.

Let X, Y, Z be the coordinates of the neighbouring point Q with position vector $\vec{r}$ then, $\vec{r}=X\vec{t}+Y\vec{n}+Z\vec{b}$.....(1)

If the curve is of class ≥4 and if s denotes the small arc length PQ.

By Taylor's theorem

$\vec{r}(s)=\vec{r}(o)+\frac{s}{1!}\vec{r}(o)+\frac{s^2}{2!}\overrightarrow{r''}(o)+\frac{s^3}{3!}\overrightarrow{r'''}(o)+\frac{s^4}{4!}\vec{r}^{iv}(o)+O(s^5)$ ...(2) as $s\to 0$

We have, $\vec{r}(o)=0,\overrightarrow{r'}(o)=\vec{t}$

$\overline{r''}=\frac{d\overline{r'}}{ds}=\frac{d}{ds}(t)=\frac{dt}{ds}=\kappa\vec{n}$

$\overline{r'''}=\frac{d\overline{r''}}{ds}=\frac{d}{ds}(\kappa\bar{n})=\kappa\frac{d\bar{n}}{ds}+\bar{n}\frac{d\kappa}{ds}$

$=\kappa(\tau\bar{b}-\kappa\bar{t})+n\kappa'$

$$\overline{r'''}=-\kappa^2\bar{t}+\kappa'\bar{n}+\kappa\tau\bar{b}$$

$\vec{r}^{iv}(o)=\frac{d}{ds}(\overrightarrow{r'''}(o))=\frac{d}{ds}(-\kappa^2\bar{t}+\kappa'\bar{n}+\kappa\tau\bar{b})$

$=\frac{d}{ds}(\kappa'\bar{n})-\frac{d}{ds}(\kappa^2\bar{t})+\frac{d}{ds}(\kappa\tau\bar{b})$

$=\kappa''^{\bar{n}}+\kappa'^{\frac{dn}{ds}}-(\kappa^2\overline{t'}+2\bar{t}\kappa\kappa')+\kappa'\tau\bar{n}+\kappa\tau'^{\bar{b}}+\kappa\tau\overline{b'}$

$=\kappa''\bar{n}+\kappa'(\tau\bar{b}-\kappa\bar{t})-\kappa^2\overline{t'}+2\kappa\kappa'\bar{t}+\kappa'\tau\bar{b}+\kappa\tau'\bar{b}+\kappa\tau\overline{b'}$

$=\kappa''^{\bar{n}}+\kappa'\tau\bar{b}-\kappa'\kappa\bar{t}-\kappa^2\overline{t'}+2\kappa\kappa'^{\bar{t}}+\kappa'\tau\bar{b}+\kappa\tau'^{\bar{b}}+\kappa\tau\overline{b'}$

$=\kappa''\bar{n}+\kappa'\tau\bar{b}-3\kappa'\kappa\bar{t}-\kappa^2(\kappa\bar{n})+\tau\kappa'\bar{b}+\kappa\tau'\bar{b}+\kappa\tau(-\tau\bar{n})$

$\vec{r}^{iv}(o)=-3\kappa'\kappa\bar{t}+(\kappa''-\kappa^2-\kappa\tau^2)\bar{n}+(\kappa'\tau+2\kappa'\tau)\bar{b}$

(2) becomes

r(s) $=0+\frac{s.(t)}{1!}+\frac{s^2}{2!}(\kappa\bar{n})+\frac{s^3}{3!}(-\kappa^2\bar{t}+\kappa'\bar{n}+\kappa\tau\bar{b})+\frac{s^4}{4!}[(-3\kappa'\kappa\bar{t}+(\kappa''-\kappa^2-\kappa\tau^2)\bar{n}+(\kappa'\tau+2\kappa'\tau)\bar{b})]$

$$\vec{r}=[s-\frac{\kappa^2s^3}{6}-\frac{\kappa\kappa's^4}{8}+O(s)]\vec{t}+[\frac{\kappa s^2}{2}+\frac{\kappa s^3}{6}+\frac{\kappa''-\kappa\tau^2-\kappa^3s^4}{24}+O(s^4)]\vec{n}+[\frac{\kappa\tau s^3}{6}+\frac{2\kappa'\tau+\kappa\tau's^4}{24}+O(s^4)]\vec{b}$$

Hence X $=s-\frac{\kappa^2s^3}{6}-\frac{\kappa\kappa's^4}{8}+O(s^4)$

Y $=\frac{\kappa s^2}{2}+\frac{\kappa s^3}{6}+\frac{\kappa''-\kappa\tau^2-\kappa^3s^4}{24}+O(s^4)$

Z $=\frac{\kappa\tau s^3}{6}+\frac{2\kappa'\tau+\kappa\tau's^4}{24}+O(s^4)$

13. Prove that

(i) $lt_{s\to 0}\frac{2Y}{X^2}=\kappa$

(ii) $lt_{s\to 0}\frac{3Z}{XY}=\tau$

(iii) $\sqrt{x^2+y^2+z^2}=s(1-\frac{\kappa^2 s^2}{24})$

Solution:

(iii) $\sqrt{x^2+y^2+z^2}=\{[s-\frac{\kappa^2 s^3}{6}-\frac{\kappa\kappa' s^4}{8}+O(s^4)]^2+[\frac{\kappa s^2}{2}+\frac{\kappa s^3}{6}+\frac{\kappa''-\kappa\tau^2-\kappa^3 s^4}{24}+O(s^4)]^2+[\frac{\kappa\tau s^3}{6}+\frac{2\kappa'\tau+\kappa\tau' s^4}{24}+O(s^4)]^2\}^{\frac{1}{2}}$

$=[s^2+\frac{s^6\kappa^4}{36}-\frac{2s^4\kappa^2}{6}+\frac{s^4\kappa^2}{4}+\frac{s^6\kappa'^2}{36}+\frac{2s^5\kappa'\kappa}{12}+\frac{s^6\kappa^2\tau^2}{36}]^{\frac{1}{2}}$

(omitting higher power)

$=[s^2-\frac{2s^4\kappa^2}{6}+\frac{s^4\kappa^2}{4}]^{\frac{1}{2}}$

$=[s^2-\frac{s^4\kappa^2}{3}+\frac{s^4\kappa^2}{4}]^{\frac{1}{2}}$

$=[s^2-\frac{s^4\kappa^2}{12}]^{\frac{1}{2}}$

$=s[1-\frac{s^2\kappa^2}{12}]^{\frac{1}{2}}$

$=[1-\frac{1}{2}(\frac{s^2\kappa^2}{12})+\frac{1}{2}(\frac{s^2\kappa^2}{12})^2-\ldots]$

$$\sqrt{x^2+y^2+z^2}=s[1-\frac{1}{2}\frac{s^2\kappa^2}{12}]$$

$$=s[1-\frac{s^2\kappa^2}{24}]$$

14. Show that the projection of the curve near 'p' on the oscillating plane is approximately the curve z=0, y $=\frac{1}{2}\boldsymbol{\kappa x^2}$ its projection on the rectifying plane is approximately y=0, z=$\frac{1}{6}\boldsymbol{\kappa\tau x^3}$ and its projection on the normal plane is approximately x=0, $\boldsymbol{z^2=\frac{2}{9}(\tau^2\kappa)y^3}$

Solution:

We know that,

'p' is any point on the curve and p(x,y,z) interms of s are

X $=s-\frac{\kappa^2 s^3}{6}-\frac{\kappa\kappa' s^4}{8}+O(s^4)$

Y $=\frac{\kappa s^2}{2}+\frac{\kappa s^3}{6}+\frac{\kappa''-\kappa\tau^2-\kappa^3 s^4}{24}+O(s^4)$

Z $=\frac{\kappa\tau s^3}{6}+\frac{2\kappa'\tau+\kappa\tau' s^4}{24}+O(s^4)$

Consider X=s, Y=$\frac{s^2\kappa}{2}$, Z= $\frac{s^3\kappa\tau}{6}$

Suppose we project the curve on the oscillating plane. (ie) z=0

Y= $\frac{s^2\kappa}{2}$ =$\frac{x^2\kappa}{2}$

Hence the curve is z=0 and y=$\frac{x^2\kappa}{2}$

Now we project the curve on the normal plane.

(ie) x=0 and z=$\frac{s^3\kappa\tau}{6}$

$$\Rightarrow z^2=\frac{s^6\kappa^2\tau^2}{36}$$

$$z^2=\frac{\tau^2}{36}(\frac{2y^3}{\kappa})=\frac{2}{9}.\frac{\tau^2}{\kappa}.y^3$$

Hence the curve is x= 0and z=$\frac{2}{9}.\frac{\tau^2}{\kappa}.y^3$

Finally, we project the curve on the rectifying plane.

(ie) y=0

z=$\frac{s^3}{6}\kappa\tau = \frac{\kappa\tau}{6}x^3$

Hence the curve is y=0 and z=$\frac{s^3}{6}\kappa\tau = \frac{\kappa\tau}{6}x^3$

Hence the curve is y=0 and z=$\frac{\kappa\tau}{6}x^3$

15. Show that the length of the common perpendicular 'd' of the tangent at two near points distance 's' about is approximately given by d =$\frac{\kappa\tau s^3}{12}$

Solution:

Let γ be any curve.

Let P,Q be any two points on the curve with parameters 'o' and 's' respectively.

$\vec{O}P = \vec{r}(o), \vec{O}Q = \vec{r}(s)$

Let the unit tangent vectors at the point P and Q be $\overrightarrow{r'}(o)$, $\overrightarrow{r'}(s)$ respectively.

(ie) $\overrightarrow{r'}(o)$ and $\overrightarrow{r'}(s)$ is common perpendicular to both $\overrightarrow{r'}(o) \times \overrightarrow{r'}(s)$

(ie) The length of the common perpendicular is

d= $\frac{[\vec{r}(s)-\vec{r}(o),\overrightarrow{r\prime}(s),\overrightarrow{r\prime}(o)]}{|\overrightarrow{r\prime}(s)\times\overrightarrow{r\prime}(o)|}$

If the curve is of class ≥ 3, then by taylor's theorem,

$\vec{r}(s) = \vec{r}(o) + \frac{s}{1!}\vec{r}(o) + \frac{s^2}{2!}\overrightarrow{r''}(o) + \frac{s^3}{3!}\overrightarrow{r'''}(o) + O(s^4)$

We know that, $\overrightarrow{r'}(o) = \vec{t}$

$\overline{r}'' = \frac{d\overline{r\prime}}{ds} = \frac{d}{ds}(t) = \frac{dt}{ds} = \kappa\vec{n}$

$$\overline{r}''' = \frac{d\overline{r}''}{ds} = \frac{d}{ds}(\kappa\overline{n})$$

$$\overline{r}''' = -\kappa^2\overline{t} + \kappa'\overline{n} + \kappa\tau\overline{b}$$

Now, r(s)-r(o) = st+$\frac{1}{2}s^2\kappa n + \frac{s^3}{6}(-\kappa^2\overline{t} + \kappa'\overline{n} + \kappa\tau\overline{b}) + O(s^3)$

Diff. ẇ.r to s, we get

$\overrightarrow{r'}(s)$=t+$\frac{1}{2}2s\kappa n + \frac{2s^2}{6}(-\kappa^2\overline{t} + \kappa'\overline{n} + \kappa\tau\overline{b}) + O(s^3)$

=t+$s\kappa n + \frac{2s^2}{2}(-\kappa^2\overline{t} + \kappa'\overline{n} + \kappa\tau\overline{b}) + O(s^3)$

=$t - \frac{1}{2}s^2\kappa^2 t + s\kappa n + \frac{1}{2}s^2\kappa' n + \frac{s^2\kappa\tau b}{2}$

=$(1 - \frac{s^2\kappa^2}{2})\overline{t} + (s\kappa + \frac{1}{2}s^2\kappa')\overline{n} + \frac{\kappa\tau s^2}{2}\overline{b}$

$$[\vec{r}(s) - \vec{r}(o), \overrightarrow{r'}(s), \overrightarrow{r'}(o)]$$

$$= \begin{vmatrix} s - \frac{\kappa^2 s^3}{6} & \frac{s^2\kappa}{2} + \frac{s^3\kappa}{6} & \frac{s^3\kappa\tau}{6} \\ 1 & 0 & 0 \\ 1 - \frac{s^2\kappa^2}{2} & s\kappa + \frac{1}{2}s^2\kappa' & \frac{\kappa\tau s^2}{2} \end{vmatrix}$$

=$(s - \frac{\kappa^2 s^3}{6})(0) - \frac{s^2\kappa}{2} + \frac{s^3\kappa}{6}(\frac{\kappa\tau s^2}{2} - 0) + \frac{s^3\kappa\tau}{6}(s\kappa + \frac{1}{2}s^2\kappa')$

=$\frac{-s^4\kappa^2\tau}{4} - \frac{s^5\kappa\kappa\prime\tau}{12} + \frac{s^4\kappa^2\tau}{6} + \frac{s^5\kappa\kappa\prime\tau}{12}$

=$s^4\kappa^2\tau[\frac{-1}{4} + \frac{1}{6}]$

NOTES

$=-s^4\kappa^2\tau[\frac{1}{4}-\frac{1}{6}]$

$[\vec{r}(s)-\vec{r}(o),\overrightarrow{r'}(s),\overrightarrow{r'}(o)]=-\frac{1}{12}s^4\kappa^2\tau$

$$\overrightarrow{r'}(o)\times\overrightarrow{r'}(s)=\begin{vmatrix}\vec{t} & \vec{n} & \vec{b}\\ 1 & 0 & 0\\ 1-\frac{s^2\kappa^2}{2} & s\kappa+\frac{1}{2}s^2\kappa' & \frac{\kappa\tau s^2}{2}\end{vmatrix}$$

$=\vec{t}(0)-\vec{n}(\frac{\kappa\tau s^2}{2}-0)+\vec{b}(s\kappa+\frac{1}{2}s^2\kappa')$

$=-\vec{n}\frac{\kappa\tau s^2}{2}+\vec{b}s\kappa+\vec{b}\frac{1}{2}s^2\kappa'$

$|\overrightarrow{r'}(o)\times\overrightarrow{r'}(s)|=\sqrt{(-\tau)^2\frac{\kappa^2\tau s^4}{2}+s^2\kappa^2+\frac{1}{4}s^4\kappa'^4}$

$=\sqrt{\tau^2\frac{\kappa^2\tau s^4}{2}+s^2\kappa^2+\frac{1}{4}s^4\kappa'^4}$

$=\sqrt{s^2\kappa^2}$ (omitting higher power)

$=s\kappa$

d= $\frac{-1}{2}\frac{s^4\kappa^2\tau}{s\kappa}=\frac{-s^3\kappa\tau}{12}$

d =$\frac{-s^3\kappa\tau}{12}$, since distance is positive.

Curvature and torsion of a curve given as the intersection of two surfaces:

Theorem 2.1

Find the curvature and torsion of a curve given as the intersection of two surfaces.

Proof:

Let the given two surfaces are f(x,y,z)=0 .....(1)
and g(x,y,z)=0 .....(2)
Also the intersection (1) and (2) is in the curve.
Let t be the unit tangent vector at a given point p.
(ie) t lies on the tangent of two surfaces at the point p.
(ie) t is parallel to the common perpendicular of the normals.
Now, we denote the normal of the finite surfaces as $\nabla f=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})and\nabla g=(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z})$
The common perpendicular to the surface is $\nabla f\times\nabla g$, (ie) $\nabla f\times\nabla g$=h (say)
(ie) h is parallel to t.
(ie) h = λt .....(3)
Now, $\lambda t=\lambda\frac{dr}{ds}$ = $\lambda\frac{d}{ds}(r)$

$\lambda t=\triangle$ .r

Put the operator $\triangle$ on both sides (3),

$$\triangle\lambda t=\triangle h$$

$$\lambda\frac{d}{ds}(\lambda t)=\triangle h$$

$$\lambda(\lambda t'+\lambda' t)=\triangle h$$

$\lambda^2\kappa\bar{n}+\lambda\lambda' t=\triangle h$ ....(4)

Taking the cross product of eqn(3) and (4)

$(\lambda t)\times(\lambda^2\kappa\bar{n}+\lambda\lambda' t)=h\times\triangle h$

$(\lambda t) \times (\lambda^2 \kappa \bar{n}) = h \times \triangle\, h$
$\lambda^3 \kappa (t \times n) = h \times \triangle\, h$
$\lambda^3 \kappa b = h \times \triangle\, h$
$\lambda^3 \kappa$ b= m .....(5) where m=$h \times \triangle\, h$
Taking modulus on both sides, we get
$|\lambda^3 \kappa$ b|= |m|
$\sqrt{(\lambda^3 \kappa b)}^2$= |m|
$\lambda^3 \kappa$= |m|, $\kappa = \frac{|m|}{\lambda^3}$ .....(6)
Put the operator $\triangle$ on both side (5), $\triangle\, (\lambda^3 \kappa b) = \triangle\, m$

$$\lambda \frac{d}{ds} (\lambda^3 \kappa b) = \triangle\, m$$
$$\lambda(\lambda^3 \kappa b' + \lambda^3 \kappa' b + 3\lambda^2 \lambda' \kappa b) = \triangle\, m$$
$$\lambda(\lambda^3 \kappa(-\tau n) + \lambda^3 \kappa' b + 3\lambda^2 \lambda' \kappa b) = \triangle\, m$$
$$(-\lambda^4 \kappa(-\tau n) + \lambda^4 \kappa' b + 3\lambda^3 \lambda' \kappa b) = \triangle\, m$$
$$-\lambda^4 \kappa(-\tau n) + (\lambda^4 \kappa' + 3\lambda^3 \lambda' \kappa) b = \triangle\, m \text{ ....(7)}$$

Taking scalar product on eqn(4) and (7)

$$(\lambda^2 \kappa \bar{n} + \lambda \lambda' t).(-\lambda^4 \kappa(-\tau n) + (\lambda^4 \kappa' + 3\lambda^3 \lambda' \kappa) b) = \triangle\, h.\triangle\, m$$
$$\lambda^2 \kappa \bar{n}. -\lambda^4 \kappa(-\tau n) = \triangle\, h.\triangle\, m$$
$$-\lambda^6 \kappa^2 \tau = \triangle\, h.\triangle\, m$$
$$-\lambda^6 \frac{|m|^2}{\lambda^6} \tau = \triangle\, h.\triangle\, m$$
$$|m|^2 \tau = \triangle\, h.\triangle\, m$$
$$\tau = \frac{\triangle h.\triangle m}{|m|^2}$$

16. Find the curvature and torsion of the curve of intersection of the two quadratic surfaces $ax^2 + by^2 + cz^2 = 1$ .....(1), $a'x^2 + b'y^2 + c'z^2 = 1$ .....(2)

Solution:

Given two surfaces are
f(x,y,z)=$\frac{1}{2}(ax^2 + by^2 + cz^2 - 1)$
g(x,y,z)=$\frac{1}{2}(a'x^2 + 'by^2 + c'z^2 - 1)$
Then ∇f=(ax, by, cz)
∇g=(a'x, b'y, c'z)

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ ax & by & cz \\ a'x & b'y & c'z \end{vmatrix}$$

$=\vec{t}(bc' - b'c)yz + \vec{n}(a'c - c'a)xz + \vec{b}(ab' - a'b)xy$
=(Ayz, Bxz, Cxy), where A=bc'-b'c, B=a'c-c'a, C=ab'-a'b
=xyz$(\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$
Put $\lambda t = \lambda r'$= $(\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$ .....(3)
Taking the scalar product (3)
$\lambda t.\lambda t = (\frac{A^2}{x^2}, \frac{B^2}{y^2}, \frac{C^2}{z^2})$
$\lambda^2 = (\frac{A^2}{x^2}, \frac{B^2}{y^2}, \frac{C^2}{z^2})$
Taking operator $\triangle$ on both sides (3)

NOTES

$\Delta\ \lambda t = \Delta\ (\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$

$\lambda \frac{d}{ds}(\lambda t) = \Delta\ (\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$

$\lambda(\lambda t' + \lambda' t) = \Delta\ (\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$

$(\lambda^2 t' + \lambda\lambda' t) = \lambda \frac{d}{ds}(\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$

$(\lambda^2 t' + \lambda\lambda' t) = \lambda(\frac{-A}{x^2}.\frac{dx}{ds}, \frac{-B}{y^2}.\frac{dy}{ds}, \frac{-C}{z^2}.\frac{dz}{ds})$ .....(4)

Now, eqn(3) becomes $\lambda(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) = (\frac{A}{x}, \frac{B}{y}, \frac{C}{z})\ \lambda\frac{dx}{ds} = \frac{A}{x} \Rightarrow \frac{dx}{ds} = \frac{A}{\lambda x}$

$$\lambda\frac{dy}{ds} = \frac{B}{y} \Rightarrow \frac{dy}{ds} = \frac{B}{\lambda y}$$

$\lambda\frac{dz}{ds} = \frac{C}{z} \Rightarrow \frac{dz}{ds} = \frac{C}{\lambda z}$ ......(5)

Sub in (4)

$$(\lambda^2 t' + \lambda\lambda' t) = \lambda(\frac{-A}{x^2}.\frac{A}{\lambda x}, \frac{-B}{y^2}.\frac{B}{\lambda y}, \frac{-C}{z^2}.\frac{C}{\lambda z})$$

$$= \lambda(\frac{-A^2}{\lambda x^3}, \frac{-B^2}{\lambda y^3}, \frac{-C^3}{\lambda z^2})$$

$(\lambda^2 \kappa n + \lambda\lambda' t) = (\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^3}{z^2})$ ....(6)

Taking the cross product of eqn(3) and (6)

$(\lambda^2 \kappa n + \lambda\lambda' t) \times \lambda t = (\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^3}{z^2}) \times (\frac{A}{x}, \frac{B}{y}, \frac{C}{z})$

$$\lambda^3 \kappa b = \begin{vmatrix} i & j & k \\ \frac{A}{x} & \frac{B}{y} & \frac{C}{z} \\ \frac{A^2}{x^3} & \frac{B^2}{y^3} & \frac{C^2}{z^3} \end{vmatrix}$$

$= i(\frac{C^2B}{z^3y} - \frac{B^2C}{zy^3}) - j(\frac{C^2A}{z^3x} - \frac{A^2C}{zx^3}) + k((\frac{B^2A}{y^3x} - \frac{A^2B}{yx^3}))$

$= i(\frac{C^2By^2 - B^2Cz^2}{z^3y^3}) - j(\frac{C^2Ax^2 - A^2Cz^2}{x^3z^3}) + k(\frac{B^2Ax^2 - A^2By^2}{x^3y^3})$

$\lambda^3 \kappa b = [\frac{BC}{yz}(\frac{Cy^2 - Bz^2}{z^2y^2}), \frac{AC}{xz}(\frac{Az^2 - Cx^2}{x^2y^2}), \frac{AB}{xy}(\frac{Bx^2 - Ay^2}{x^2y^2})]$ .....(7)

Eqn(1)$\times a' - (2) \times a$

$\Rightarrow (aa'x^2 + ba'y^2 + ca'z^2 - a') - (a'ax^2 + b'ay^2 + c'az^2 - a) = aa'x^2 + ba'y^2 + ca'z^2 - a' - a'ax^2 - b'ay^2 - c'az^2 + a$

$\Rightarrow (ba' - b'a)y^2 + (ca' - c'a)z^2 + (a - a') = 0 \Rightarrow (a'b - ab')y^2 + (a'c - ac')z^2 = a' - a$, since B=a'c-ac', C=a'b-ab'

$$\Rightarrow -cy^2 + Bz^2 = a - a'$$

$\Rightarrow cy^2 - Bz^2 = a' - a$ ....(8a)

(1)$\times b' - (2) \times b$

$\Rightarrow (ab'x^2 + bb'y^2 + cb'z^2 - b') - (a'bx^2 + b'by^2 + c'bz^2 - b) = ab'x^2 + bb'y^2 + cb'z^2 - b' - a'bx^2 - b'by^2 - c'bz^2 + b$

$$\Rightarrow (ab' - a'b)x^2 + (cb' - c'b)z^2 - (b' - b) = 0$$

$$\Rightarrow (ab' - a'b)x^2 + (cb' - c'b)z^2 = (b' - b)$$

$$\Rightarrow (ab' - a'b)x^2 - (c'b - cb')z^2 = (b' - b)$$

$$\Rightarrow Cx^2 - Az^2 = b' - b$$

$\Rightarrow Az^2 - Cx^2 = b - b'$ ....(8b)

(1)$\times c' - (2) \times c$

$\Rightarrow (ac'x^2 + bc'y^2 + cc'z^2 - c') - (a'cx^2 + b'cy^2 + c'cz^2 - c)$

$$\Rightarrow (ac' - a'c)x^2 + (bc' - b'c)y^2 - (c' - c)$$

$$\Rightarrow (ac' - a'c)x^2 + (bc' - b'c)y^2 = (c - c')$$
$\Rightarrow Bx^2 - Ay^2 = c - c'$ ....(8c)

Sub 8a, 8b, 8c in(7),

$\lambda^3\kappa b$=$[\frac{BC}{yz}(\frac{a-a'}{z^2y^2}), \frac{AC}{xz}(\frac{b-b'}{x^2y^2}), \frac{AB}{xy}(\frac{c-c'}{x^2y^2})]$

$\lambda^3\kappa b$=$\frac{ABC}{x^3y^3z^3}[\frac{x^3(a-a')}{A}, \frac{y^3(b-b')}{B}, \frac{z^3(c-c')}{C}]$ ....(9)

Put $\mu\bar{b}$=$[\frac{x^3(a-a')}{A}, \frac{y^3(b-b')}{B}, \frac{z^3(c-c')}{C}]$ ....(10)

Taking the scalar product of eqn(10) itself

$\mu\bar{b}.\mu\bar{b}$=$[\frac{x^6(a-a')^2}{A^2}, \frac{y^6(b-b')^2}{B^2}, \frac{z^6(c-c')^2}{C^2}]$

$$\mu^2 = \sum \frac{x^6(a-a')^2}{A^2}$$

Taking the scalar product of eqn(9) itself

$\lambda^3\kappa b.\lambda^3\kappa b$=$\frac{A^2B^2C^2}{x^6y^6z^6}[\frac{x^6(a-a')^2}{A^2}, \frac{y^6(b-b')^2}{B^2}, \frac{z^6(c-c')^2}{C^2}]$

$\lambda^6\kappa^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\sum \frac{x^6(a-a')^2}{A^2}$ .....(11)

$\lambda^6\kappa^2 = \frac{A^2B^2C^2}{x^6y^6z^6}\mu^2$ .....(12)

$$\kappa^2 = \frac{A^2B^2C^2}{x^6y^6z^6}.\frac{\mu^2}{\lambda^6}$$

From eqn(12)

$\mu^2 = \lambda^6\kappa^2\frac{x^6y^6z^6}{A^2B^2C^2}$

$\mu = \lambda^3\kappa\frac{x^3y^3z^3}{ABC}$ ....(13)

Diff. eqn(10) w.r to s,

$\mu b' + \mu' b$=$[3x^2.\frac{dx}{ds}\frac{(a-a')}{A}, 3y^2.\frac{dy}{ds}\frac{(b-b')}{B}, 3z^2.\frac{dz}{ds}\frac{(c-c')}{C}]$

Sub in (5) in(10)

$-\tau n\mu + \mu' b = \frac{3}{\lambda}[x(a-a'), y(b-b'), z(c-c')]$ .....(14)

Taking scalar product of eqn (6) and (14)

$$(\lambda^2\kappa n + \lambda\lambda' t).(-\tau n\mu + \mu' b) = (\frac{-A^2}{x^3}.\sum \frac{3}{\lambda}x(a-a')$$
$$\Rightarrow \lambda^2\kappa\mu\tau = (\frac{-A^2}{x^3}.\sum \frac{3}{\lambda}x(a-a')$$
$$\tau = \frac{3\sum A^2(a-a')}{\lambda^2x^3\kappa\mu}$$
$$= \frac{3\sum \frac{A^2}{x^2}(a-a')}{\lambda^2\kappa.\frac{\lambda^2\kappa x^3y^3z^3}{ABC}}$$
$$= \frac{3ABC\sum \frac{A^2}{x^2}(a-a')}{x^3y^3z^3[\frac{A^2B^2C^2}{x^6y^6z^6}.\sum \frac{x^6}{A^2}(a-a')^2]}$$
$$\tau = \frac{3x^3y^3z^3\sum \frac{A^2}{x^2}(a-a')}{ABC\sum \frac{x^6}{A^2}(a-a')^2}$$

2.4 Check your progress

1. Define curvature
2. Define torsion

3. Write Serret-Frenet formula

2.6 Summary

The arc rate at which the tangent changed direction as p moves along the curve is called curvature of the curve and its denoted by κ.
By definition $|\kappa| = |t'|$
As p moves along the curve the arc rate at which the oscillating plane turns about the tangent is called the torsion of a curve and its denoted by τ
Serret-Frenet formula:

(i) Prove that $\frac{d\bar{t}}{ds} = \kappa\bar{n}$
(ii) $\frac{d\bar{n}}{ds} = \tau\bar{b} - \kappa\bar{t}$
(iii)$\frac{d\bar{b}}{ds} = -\tau\bar{n}$

The necessary and sufficient that a curve to be a straight line is that κ=0 at all points.
The necessary and sufficient condition that a curve γ be a plane curve is that τ=0 at all points.
The necessary and sufficient condition for the curve to be the plane curve is $[\overrightarrow{r'}, \overrightarrow{r''}, \overrightarrow{r'''}]=0$
The necessary and sufficient condition that a curve to ba a plane is $[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]=0$

2.7 Keywords

Curvature: $\kappa=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}{|\dot{\bar{r}}|^3}$ (Curvature), $\boldsymbol{k} = |\overline{\boldsymbol{r}'} \times \overline{\boldsymbol{r}''}|=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}^3|}$

Torsion: $\tau=\frac{[\dot{\bar{r}},\ddot{\bar{r}},\dddot{\bar{r}}]}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}$ (Torsion), $\boldsymbol{\tau} = \frac{[\overrightarrow{r'},\overrightarrow{r''},\overrightarrow{r'''}]}{|\overrightarrow{r'}\times\overrightarrow{r''}|^2} = \frac{[\vec{\dot{r}},\vec{\ddot{r}},\vec{\dddot{r}}]}{|\vec{\dot{r}}\times\vec{\ddot{r}}|^2}$

Coordinates of a point interms of s:

X $=s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa' s^4}{8} + O(s^4)$
Y $=\frac{\kappa s^2}{2} + \frac{\kappa s^3}{6} + \frac{\kappa'' - \kappa\tau^2 - \kappa^3 s^4}{24} + O(s^4)$
Z $=\frac{\kappa\tau s^3}{6} + \frac{2\kappa'\tau + \kappa\tau' s^4}{24} + O(s^4)$

2.8 Self Assessment Questions and Exercises

1. Find the curvature and torsion of the curves
i) r=(u, $\frac{1+u}{u}$, $\frac{1-u^2}{u}$) ii) r=(3u, $3u^2$, $2u^3$)
iii) r=(cos 2u, sin 2u, 2 sin u) iv) r=(4a $\cos^3$u, 4a $\sin^3$u, 3a cos 2u)

2. Find the curvature and the torsion of the curves given by
a) r=[a(3u-u^2), 3au^2, a$(3u + u^3)$]
b) r=[a(u-sinu), a(1-cosu), bu]
c) r=[a(1+cosu), asinu, 2asin$\frac{1}{2}$ u]

3. Prove the following relations κτ
i) r'.r''=0 ii) r'.r'''=-κ^2
iii) r'.r'''=-3κκ' iv) r'.r'''=κκ'
v) r'.r'''=κ(κ''+κ^3-κτ^2) vii) r'.r'''=κ' κ''+2κ^2 κ'+κ^2ττ'+κκ'τ^2

4. Prove that if the principal normals of a curve are binormal of another curve, then $a(\kappa^2+\tau^2)=b\kappa$ where a and b are constants.
5. Show that the angle between the principal normals at O and P is $s(\kappa^2 + \tau^2)^{1/2}$ where s is the atcual distance between O and P.
6. Show that the unit principal normal and unit binormal of the involutes of a curve C are
$n_1=\frac{\tau b-\kappa t}{\kappa\kappa_{1(c-s)}}$, $b_1=\frac{\tau t+\kappa b}{\kappa\kappa_{1(c-s)}}$.

2.9 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

UNIT-III CONTACT BETWEEN CURVES AND SURFACES

Structure

3.1 Introduction

In this unit we shall establish the conditions for the contact of curves and surfaces leading to the definitions of osculating circle and osculating sphere at a point on the space curve and also the evolutes and involutes. Before concluding this chapter, we can explain the radius of curvature, centre of curvature and tangent surfaces and their uses.

3.2 Objectives

After going through this unit, you will be able to:

- Define osculating circle and osculating sphere
- Derive the properties of osculating sphere and osculating circle
- Derive the equations of involutes and evolutes
- Find the equations of radius of curvature, centre of curvature
- Solve the problems in contact between curves and surfaces
- Find the conditions of tangent surfaces

3.3 Contact between curves and surfaces

Let F(x,y,z) =0 ....(1) be any surfaces.
Let γ be any curve denoted by $\vec{r}(f(u), g(u), h(u))$ ....(2)
The point (f(u),g(u),h(u)) lies on the surface F(x,y,z) =0
(ie) x =f(u), y =g(u), z =h(u)
The surface F(f(u),g(u),h(u)) =0
(ie) F(u) =0
If u_0 is a zero of F(u)=0 then F(u) can be expressed by the Taylor's theorem,

F(u)= $\varepsilon F'(u_0) + \frac{\varepsilon^2}{2!}F''(u_0) + \ldots + \frac{\varepsilon^n}{n!}F^{(n)}(u_0) + O(\varepsilon^{n+1})$, where $\varepsilon = u - u_0$

Then the following cases arise
(i) If F'$(u_0) \neq 0$ then the curve has one point contact with the surface.

(ii) The curve and surface intersecting at one point.
(iii) If F'$(u_0) = 0$ and F''$(u_0) \neq 0$ then the curve has 2 points contact with the surface.
(Proceeding like this) In general, if F'(u_0)=F''(u_0)=....=F$^{(n-1)}(u_0)$=0 and F$^{n}(u_0) \neq 0$ then the curve has n-points contact with the surface.

Osculating sphere:

The osculating sphere at a point on a curve is the sphere which has four points contact with the curve at p.

Theorem 2.2

Find the oscillating sphere at p(s=0) to the given curve $\vec{r} = \vec{r}(s)$.

Proof:

Let $\vec{o}$ be the position vector of the center 'c' and radius R of the osculating sphere.
Then the equation of the sphere is $(\vec{c} - \vec{r})^2 = R^2$
The point of intersecting with the curve is given by
F(s) =$(\vec{c} - \vec{r})^2 - R^2$=0, since the sphere has four points contact, we have,
F(0)=F'(0)=F''(0)=F'''(0)=0
F(0)=0 ⇒ $(\vec{c} - \vec{r})^2 = R^2$ ....(1)
F'(0)=0 ⇒ $2(\vec{c} - \vec{r})(-\vec{r'}) = 0$
⇒ $(\vec{c} - \vec{r}).t = 0$ ....(2)
F''(0)=0 ⇒ $(\vec{c} - \vec{r}).t' + t.(-\vec{r'}) = 0$
⇒ $t'(\vec{c} - \vec{r}) - \vec{r'}t = 0$
⇒ $(\vec{c} - \vec{r})k\vec{n} - t.t = 0$
⇒ $(\vec{c} - \vec{r})k\vec{n} - 1 = 0$
⇒ $(\vec{c} - \vec{r})k\vec{n} = 1$ .....(3)

$$\Rightarrow (\vec{c} - \vec{r}).\vec{n} = \frac{1}{k}$$

F''(0)=0

$$\Rightarrow (\vec{c} - \vec{r})k\vec{n'} + (\vec{c} - \vec{r})k'\vec{n} + k\vec{n}(-\vec{r'}) = 0$$
$$\Rightarrow (\vec{c} - \vec{r})k(\tau\vec{b} - k\vec{t}) + (\vec{c} - \vec{r})k'\vec{n} - k\vec{n}t = 0$$
$$\Rightarrow k\tau(\vec{c} - \vec{r})\vec{b} - k^2(\vec{c} - \vec{r})\vec{t} + k'(\vec{c} - \vec{r})\vec{n}) - k\vec{n}t = 0$$
$$\Rightarrow k\tau(\vec{c} - \vec{r})\vec{b} + k'.\frac{1}{k} = 0$$

We have, k =$\frac{1}{\rho}$, k' =$\frac{-1}{\rho^2}\rho'$, $\sigma = \frac{1}{\tau}$, $\tau = \frac{1}{\sigma}$
(4) ⇒ $(\vec{c} - \vec{r})\frac{1}{\rho}.\frac{1}{\sigma}\bar{b} + (\frac{-\rho'}{\rho^2})\rho$=0

$$\Rightarrow \frac{(\vec{c} - \vec{r})\vec{b}}{\sigma\rho} = \frac{\rho'}{\rho}$$

⇒ $(\vec{c} - \vec{r})\vec{b} = \sigma\rho'$ .....(5)
(2) ⇒ $(\vec{c} - \vec{r}) is perpendicular to \vec{t}$
⇒ lies on the normal plane.
⇒ $(\vec{c} - \vec{r}) = \lambda\vec{n} + \mu\vec{b}$ .....(6)
Sub. ėqn (6) in (5)
$(\lambda\vec{n} + \mu\vec{b})\vec{b} = \sigma\rho' \Rightarrow \mu = \sigma\rho'$
Sub. ėqn (6) in (3)
$(\vec{c} - \vec{r})k\vec{n}$=1
$(\lambda\vec{n} + \mu\vec{b})k\vec{n}$=1 ⇒ $k\lambda = 1 \Rightarrow \lambda = \frac{1}{k}$
(ie) $\lambda = \rho$

(6) ⇒ $(\vec{c}-\vec{r}) = \rho\vec{n} + \rho'\sigma\vec{b}$ ....(7)
The center of the osculating sphere is $\vec{c} = \vec{r} + \rho\vec{n} + \rho'\sigma\vec{b}$
Squaring on both sides, we have $(\vec{c}-\vec{r})^2 = (\rho\vec{n} + \rho'\sigma\vec{b})^2$

$$R^2 = \rho^2 + \rho'^2\sigma^2$$

R=$\sqrt{\rho^2 + \rho'^2\sigma^2}$ is the radius of the osculating sphere.

Corollary 2.3 *If k is constant then R=ρ*
Proof:
Given k is constant
⇒ $\frac{1}{k}$ is constant.
⇒ ρ is constant.
⇒ ρ'=0
Let $\vec{c} = \vec{r} + \rho\vec{n} + \rho'\sigma\vec{b}$ be the centre of the osculating plane.
$\vec{c} = \vec{r} + \rho\vec{n}$ and R=$\sqrt{\rho^2 + \rho'^2\sigma^2}$
R = ρ
1. Find the locus of the centre of the spherical curvature.
Solution:
Let 'c' be the original curve and c $_1$ be the locus center of the spherical curvature.
The position vector $\vec{r}$ of the centre of spherical curvature is given by,
$\vec{r_1} = \vec{r} + \rho\vec{n} + \rho'\sigma b$ ....(1)
Diff. w.r to s,

$$\frac{dr_1}{ds} = \overrightarrow{r'} + \rho\overrightarrow{n'} + \rho'\vec{n} + \rho'\sigma\overrightarrow{b'} + \rho''\sigma\vec{b} + \rho'\vec{b}\sigma'$$

$$\frac{dr_1}{ds}.\frac{dr_1}{ds} = \vec{t} + \rho(\tau\vec{b} - k\vec{t}) + \rho'\vec{n} + \rho'\sigma'\vec{b} + \rho''\sigma\vec{b} + \rho'(-\tau\vec{n})\sigma'$$

$t_1 s'_1 = \frac{\rho}{\sigma}\vec{b} + \rho''\sigma\vec{b} + \rho'\sigma'\vec{b}$ ......(2), since s_1 is an increasing function of s, so that s'_1 is non-negative.
We write t $_1$ =eb, where e=± 1
$t_1.t_1 s'_1 = (\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')t_1.b$
$s'_1 = (\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')b.eb$
$s'_1 = (\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')e$ .....(3) since t $_1$ =eb .....(4)
Diff(4) w.r to s, $\frac{dt_1}{ds} = e.\frac{db}{ds}$

$$\frac{dt_1}{ds_1}.\frac{ds_1}{ds} = e(-tau\bar{n})$$

$$t'_1.s'_1 = -e\tau\bar{n}$$

$$(k_1 n_1).s'_1 = -e\tau\bar{n}$$

Put $\bar{n}_1 = e_1\bar{n}$ where $e_1 = \pm 1$
$k_1 e_1\bar{n}.s'_1 = -e\tau\bar{n}$ $k_1 e_1 s'_1 = -e\tau$

$$k_1^2 s'^2_1 = \tau^2 \Rightarrow\ k_1^2 = \frac{\tau^2}{s'^2_1}$$

$$\Rightarrow\ k_1 = \pm\frac{\tau}{s'_1} \Rightarrow k_1 = \frac{\tau e}{s'_1}$$

$$\Rightarrow k_1 = \frac{\tau e}{(\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')e}$$

$$\Rightarrow\ k_1 = \frac{\tau e}{(\frac{\rho + \rho''\sigma^2 + \rho'\sigma'\sigma}{\sigma})e}$$

$$\Rightarrow k_1 = \frac{\tau e}{(\frac{\rho + \rho''\sigma^2 + \rho'\sigma'\sigma}{\frac{1}{\tau}})e}$$

$$\Rightarrow k_1 = \frac{1}{\rho + \rho''\sigma^2 + \rho'\sigma'\sigma}$$

$\Rightarrow k_1 = (\rho + \rho''\sigma^2 + \rho'\sigma'\sigma)^{-1}$ ....(5)

The unit binormal vector b_1 is parallel to $\bar{t}$.

Since, $\quad b_1 = t_1 \times n_1 = eb \times e_1 n$

$= ee_1(b \times n) = -ee_1 t$

$b_1 = -ee_1 t$

Diffẇ.r to s,

$$\frac{db_1}{ds} = -ee_1\frac{dt}{ds}$$

$$\frac{db_1}{ds_1} \times \frac{ds_1}{ds} = -ee_1 t'$$

$$b'_1 . s'_1 = -ee_1 kn$$

$$\Rightarrow -\tau_1 n_1 s'_1 = -ee_1 kn$$

$$\Rightarrow \tau_1 n_1 s'_1 = \frac{ee_1 kn}{n_1}$$

$$\Rightarrow \tau_1 n_1 s'_1 = \frac{ee_1 kn}{e_1 n} n_1 = e_1 n$$

$$\Rightarrow \tau_1 n_1 s'_1 = ek$$

$$\Rightarrow \tau_1 = \frac{ek}{s'_1}$$

$$\Rightarrow \tau_1 = \frac{ek}{(\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')e}$$

$\Rightarrow \tau_1 = \frac{1}{(\frac{\rho}{\sigma}+\rho''\sigma+\rho'\sigma')\rho}$ ....(6) k=$\frac{1}{\rho}$

Eqn (5) and (6) given the curvature and torsion of the curve e_1 at a point p corresponding to the curve c.

$$\frac{(5)}{(6)} \Rightarrow \frac{k_1}{\tau_1} = \frac{(\rho + \rho''\sigma^2 + \rho'\sigma'\sigma)^{-1}}{\frac{1}{(\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma')\rho}}$$

$=(\frac{1}{\rho+\rho''\sigma^2+\rho'\sigma'\sigma}) \times \rho(\frac{\rho+\rho''\sigma^2+\rho'\sigma'\sigma}{\sigma})$

$$\Rightarrow \frac{k_1}{\tau_1} = \frac{\rho}{\sigma}$$

$$\Rightarrow \frac{k_1}{\tau_1} = \frac{\tau}{k}$$

$$kk_1 = \tau\tau_1$$

2. Prove that the radius of curvature of the locus of the centre of curvature of a curve is given

$$[(\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \frac{1}{R})^2 + \frac{\rho'\sigma^4}{\rho^2 R^4}]^{-\frac{1}{2}}$$

Solution:

Let c be the given curvature and c $_1$ be the locus of the centre of curvature.

The position vector of the centre of curvature is $r_1 = \bar{r} + \rho\bar{n}$ ....(1)

Diff (1) w.r to (3),

$$\frac{dr_1}{ds} = \frac{dr}{ds} + \rho.\frac{dn}{ds} + \rho\overrightarrow{n'}$$

$$\frac{dr_1}{ds}.\frac{dr_1}{ds} = \overrightarrow{r'} + \rho(\tau\vec{b} - k\vec{t}) + \rho'\vec{n}$$

$\vec{r}_1 s'_1 = \vec{t} + \frac{\rho}{\sigma}\vec{b} - \vec{t} + \rho' n$

$\vec{t}_1 s'_1 = \frac{\rho}{\sigma}\vec{b} + \rho' n$

Multiplying both sides by $\frac{\sigma}{\rho}$

$\frac{\sigma}{\rho}\vec{t}_1 s'_1 = \vec{b} + \frac{\sigma}{\rho}\rho' n$ ....(2)

Taking scalar product on both sides,

$$\frac{\sigma^2}{\rho^2}.s'^2_1 = 1 + \frac{\sigma^2\rho'^2}{\rho^2}$$

$$\frac{\sigma^2}{\rho^2}.s'^2_1 = \frac{\rho^2 + \sigma^2\rho'^2}{\rho^2}$$

$$\Rightarrow \sigma^2.s'^2_1 = \rho^2 + \sigma^2\rho'^2$$

$$\Rightarrow s'^2_1 = \frac{R^2}{\sigma^2}$$

$s'_1 = \frac{R}{\sigma}$ ....(3)

Diff (2) w,r to s,

$$\frac{\sigma}{\rho}.s'_1\frac{dt_1}{ds} + \vec{t}_1 \times \frac{d}{ds}(\frac{\sigma}{\rho}.s'_1) = \overrightarrow{b'} + \frac{\sigma}{\rho}.\rho'\overrightarrow{n'} + n\frac{d}{ds}\frac{\sigma\rho'}{\rho}$$

$$\Rightarrow \frac{\sigma}{\rho}.s'_1\frac{dt_1}{ds_1}.\frac{ds_1}{ds} + \vec{t}_1.\frac{d}{ds}(\frac{\sigma}{\rho}.s'_1) = -\tau\bar{n} + \frac{\sigma}{\rho}.\rho'(\tau\bar{b} - k\bar{t}) + n\frac{d}{ds}\frac{\sigma\rho'}{\rho}$$

$$\Rightarrow \frac{\sigma}{\rho}.s'^2_1 k_1\bar{n}_1 + \vec{t}_1.\frac{d}{ds}(\frac{\sigma}{\rho}.s'_1) = (\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \tau)\bar{n} + \frac{\sigma}{\rho}.\rho'\tau\bar{b} - \frac{\sigma}{\rho}.\rho' k\bar{t}$$

$\Rightarrow \frac{\sigma}{\rho}.s'^2_1 k_1\bar{n}_1 + \vec{t}_1.\frac{d}{ds}(\frac{\sigma}{\rho}.s'_1) = (\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \tau)\bar{n} + \frac{\rho'}{\rho}.\bar{b} - \frac{\sigma}{\rho^2}.\rho'\bar{t}$ ....(4)

Taking cross product (2) and (4)

$$\frac{\sigma^2}{\rho^2}.s'^3_1 k_1\bar{b}_1 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 0 & \frac{\sigma\rho'}{\rho} & 1 \\ -\frac{\sigma}{\rho^2}.\rho' & \frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \tau & \frac{\rho'}{\rho} \end{vmatrix}$$

$=\vec{t}[\frac{\sigma}{\rho^2}.\rho'^2 - \frac{d}{ds}(\frac{\sigma\rho'}{\rho}) + \tau] + \vec{n}(\frac{\sigma}{\rho^2}.\rho') + \vec{b}(\frac{\sigma^2}{\rho^3}.\rho'^2)$

Taking scalar product on both sides,

$$\frac{\sigma^4}{\rho^4}.s'^6_1 k_1^2 = [\frac{\sigma}{\rho^2}.\rho'^2 - \frac{d}{ds}(\frac{\sigma\rho'}{\rho}) + \tau]^2 + (\frac{\sigma^2}{\rho^4}.\rho'^2) + (\frac{\sigma^4}{\rho^6}.\rho'^4)$$

$$\frac{\sigma^4}{\rho^4}.(\frac{R}{\sigma})^6 k_1^2 = [\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \frac{\sigma}{\rho^2}.\rho'^2 - \tau]^2 + (\frac{\sigma^2}{\rho^4}.\rho'^2) + (\frac{\sigma^4}{\rho^6}.\rho'^4)$$

$$\frac{R^6}{\rho^4\sigma^2}.\frac{1}{)}R_1^2 = [\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \frac{\sigma}{\rho^2}.\rho'^2 - \tau]^2 + (\frac{\sigma^2\rho'^2\rho^2 + \sigma^4\rho'^4}{\rho^6}$$

$$\frac{1}{R_1^2} = \frac{\rho^4\sigma^2}{R^6}\{[\frac{d}{ds}(\frac{\sigma\rho'}{\rho}) - \frac{\sigma}{\rho^2}.\rho'^2 - \tau]^2 + (\frac{\sigma^2\rho'^2(\rho^2 + \sigma^2\rho'^2)}{\rho^6})\}$$

$=\frac{\rho^4\sigma^2}{R^6}\{[\frac{d}{ds}(\frac{\sigma\rho\prime}{\rho})-\frac{\sigma}{\rho^2}.\rho'^2-\tau]^2+(\frac{\sigma^4\rho\prime^2}{\rho^2R^4})\}$

Since $\rho=R, R^2=\rho^2+\sigma^2\rho'^2, \frac{1}{\rho}=\frac{1}{R}, \tau=\frac{1}{\rho}$

$=\frac{\rho^2\sigma}{R^3}\{[\frac{d}{ds}(\frac{\sigma\rho\prime}{\rho})-\frac{\sigma}{\rho^2}.\rho'^2-\tau]^2+(\frac{\sigma^4\rho\prime^2}{\rho^2R^4})\}$

$=[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho\prime}{\rho})-\frac{\rho^2\sigma}{R^3}[\frac{\sigma}{\rho^2}.\rho'^2+\frac{1}{\sigma}]^2+(\frac{\sigma^4\rho\prime^2}{\rho^2R^4})$

$=[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho\prime}{\rho})-\frac{\rho^2\sigma}{R^3}[\frac{\sigma^2\rho\prime^2+\rho^2}{\sigma\rho^2}]^2+(\frac{\sigma^4\rho\prime^2}{\rho^2R^4})$

$$\frac{1}{R_1^2}=[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho'}{\rho})-\frac{\rho^2\sigma}{R^3}\frac{R^2}{\sigma\rho^2}]^2+(\frac{\sigma^4\rho'^2}{\rho^2R^4})$$

$$R_1^2=\{[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho'}{\rho})-\frac{\rho^2\sigma}{R^3}\frac{R^2}{\sigma\rho^2}]^2+(\frac{\sigma^4\rho'^2}{\rho^2R^4}\})^{-1}$$

$$R_1=\{[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho'}{\rho})-\frac{1}{R}+(\frac{\sigma^4\rho'^2}{\rho^2R^4}\})^{\frac{-1}{2}}$$

Hence $\rho_1=\{[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho\prime}{\rho})-\frac{1}{R}+(\frac{\sigma^4\rho\prime^2}{\rho^2R^4}\})^{\frac{-1}{2}}$

3. Show that the osculating plane at p has 3 points contact with the curve.

Solution:

Let p be any point on the curve.

We know that,

The equation of the osculating plane is

F(s)= $[\bar{r}(s)-\bar{r}(o),\overline{r'}(o),\overline{r''}(o)]$ ....(1), where $\bar{r}(o)$ is the position vector of the point p(s=0) and s is measured from p.

Let the curve be of class ≥ 3

Then by taylor's theorem,

$\vec{r}(s)=\vec{r}(o)+\frac{s}{1!}\vec{r}(o)+\frac{s^2}{2!}\overrightarrow{r''}(o)+\frac{s^3}{3!}\overrightarrow{r'''}(o)+O(s^4)$

We know that, $\overrightarrow{r'}(o)=\vec{t}$

$\overline{r''}=\frac{d\overline{r\prime}}{ds}=\frac{d}{ds}(t)=\frac{dt}{ds}=\kappa\vec{n}$

$$\overline{r'''}=\frac{d\overline{r''}}{ds}=\frac{d}{ds}(\kappa\bar{n})$$

$$\overline{r'''}=-\kappa^2\bar{t}+\kappa'\bar{n}+\kappa\tau\bar{b}$$

Now, r(s)-r(o) = st+$\frac{1}{2}s^2\kappa n+\frac{s^3}{6}(-\kappa^2\bar{t}+\kappa'\bar{n}+\kappa\tau\bar{b})+O(s^3)$

$[\vec{r}(s)-\vec{r}(o),\overrightarrow{r'}(o),\overrightarrow{r''}(o)]=\begin{vmatrix}\frac{s}{1!} & \frac{s^2}{2!}+\frac{s^3\kappa\prime}{3!} & \frac{s^3\kappa\tau}{3!}\\ 1 & 0 & 0\\ 0 & s\kappa+\kappa & 0\end{vmatrix}$

$=s(o)+0+\frac{s^3\tau\kappa\kappa}{3!}$

$=\frac{s^3\tau\kappa^2}{6}$

Osculating circle:

Osculating circle at any point p on a curve is a circle which has three points contact with the curve at the point p. It is also known as circle of curvature.

Theorem 2.4 *Find the equation of the osculating circle at a point p to the curve $\vec{\gamma}$.*

Proof:

Let ρ be the radius and τ be the position vector of the centre of the osculating circle at p.

Clearly, the circle lies in the osculating plane at the point $\bar{p}$.

It is the section of the osculating plane with the sphere.

$(\bar{c}-\bar{r})^2=\rho^2$ ....(1)

Since the circle has 3 points contact, we have

f(0)=0, f'(0)=0, f''(0)=0, where F(s)=$(\bar{c}-\bar{r})^2=\rho^2 \Rightarrow F(s)=(\bar{c}-\bar{r})^2-\rho^2$

$$(\bar{c}-\bar{r})^2=\rho^2$$

F'(0)=0 $\Rightarrow 2(\bar{c}-\bar{r})^2.(-\overrightarrow{r'})=0$

$\Rightarrow \overrightarrow{r'}.(\bar{c}-\bar{r})$=0

$\Rightarrow \vec{t}.(\bar{c}-\bar{r})$=0

$\Rightarrow (\bar{c}-\bar{r})$ is perpendicular to t.

$\Rightarrow (\bar{c}-\bar{r})$ lies in the normal plane.

But $(\bar{c}-\bar{r})$ lies in the osculating plane. So $(\bar{c}-\bar{r})$ is along $\bar{n}$.

$(\bar{c}-\bar{r})=\rho\bar{n}$ ...(2)

(ie) c=r+$\rho\bar{n}$

F''(0)=0 $\Rightarrow (\bar{c}-\bar{r}\overrightarrow{t'}+(-r')t=0$

$$\Rightarrow (\bar{c}-\bar{r}k\vec{n}-t.t=0$$

$$\Rightarrow (\bar{c}-\bar{r}k\vec{n}-1=0$$

$$\Rightarrow (\bar{c}-\bar{r}k\vec{n}=1$$

$$\Rightarrow (\bar{c}-\bar{r}\vec{n}=\frac{1}{k}$$

$$\Rightarrow \rho\bar{n}.\bar{n}=\frac{1}{k}$$

$\Rightarrow \rho=\frac{1}{k}$ ....(3)

eqn (2) and (3) are centre and radius of osculating plane.

4. Find the coordinate of centre of spherical curvature given by the curve $\vec{r}=(acosu, asinu, acos2u)$

solution:

Given $\vec{r}=(acosu, asinu, acos2u)$

Let the centre of the osculating sphere be (α,β,γ)

we know that,

The curve has 4 point contact with the osculating sphere.

F(o)=0, $\dot{F}(o)=0, \ddot{F}(o)=0, \dddot{F}(o)=0$

The equation of the osculating sphere is $(\bar{c}-\bar{r})^2=R^2$

Take F(o)=$(\bar{c}-\bar{r})^2=R^2$

Now, F(o)=0$\Rightarrow (\bar{c}-\bar{r})^2-R^2=0$

$(\bar{c}-\bar{r})^2=R^2$ .....(1)

$\dot{F}(o)=0 \Rightarrow 2(\bar{c}-\bar{r})(-\dot{\bar{r}})=0$

$\Rightarrow (\bar{c}-\bar{r})(\dot{\bar{r}})=0$ ....(2)

$$\ddot{F}(o)=0 \Rightarrow (\bar{c}-\bar{r})(\ddot{\bar{r}})+(\dot{\bar{r}})(-\dot{\bar{r}})=0$$

$\Rightarrow (\bar{c}-\bar{r})(\ddot{\bar{r}})-(\dot{\bar{r}})^2=0$ ....(3)

$$\dddot{F}(o)=0 \Rightarrow (\bar{c}-\bar{r})(\dddot{\bar{r}})+\ddot{\bar{r}}(-\dot{\bar{r}})-2\dot{\bar{r}}\ddot{\bar{r}}=0$$

$\Rightarrow (\bar{c}-\bar{r})(\dddot{\bar{r}})-3\dot{\bar{r}}\ddot{\bar{r}}=0$ ...(4)

Given $\vec{r}=(acosu, asinu, acos2u)$

$$\dot{\bar{r}} = (-asinu, acosu, -2asin2u)$$
$$\ddot{\bar{r}} = (-acosu, -asinu, -4acos2u)$$
$$\dddot{\bar{r}} = (asinu, -acosu, 8asin2u)$$
$$\dot{F}(o) = 0 \Rightarrow (\bar{c} - \bar{r})(\dot{\bar{r}}) = 0$$
$$(\alpha - acosu, \beta - asinu, \gamma - acos2u)(-asinu, acosu, -2asin2u) = 0$$
$$\Rightarrow (\alpha - acosu)(-asinu) + (\beta - asinu)(acosu) + (\gamma - acos2u)(-2asin2u) = 0$$
$$\Rightarrow -\alpha asinu + a^2cosusinu + \beta acosu - a^2cosusinu - 2a\gamma sin2u + 2a^2sin2ucos2u = 0$$
$$\Rightarrow -\alpha asinu + \beta acosu - 2a\gamma sin2u + 2a^2sin2ucos2u = 0$$
$$\Rightarrow -\alpha asinu + \beta acosu - 2a\gamma sin2u + a^2sin4u = 0$$
$$\Rightarrow -a(\alpha sinu - \beta cosu + 2\gamma sin2u - asin4u) = 0$$
$$\Rightarrow \alpha sinu - \beta cosu + 2\gamma sin2u - asin4u = 0 \text{ ....(5)}$$
$\ddot{F}(o) = 0 \Rightarrow (\bar{c} - \bar{r})(\ddot{\bar{r}}) - (\dot{\bar{r}})^2 = 0$ $[(\alpha - acosu), (\beta - asinu), (\gamma - acos2u)][(-acosu, -asinu, -4acos2u)] - [(asinu, -acosu, 8asin2u)] = 0$

$\Rightarrow -\alpha acosu + a^2cos^2u - a\beta sinu + a^2sin^2u - 4a\gamma cos2u + 4a^2cos^22u - a^2sin^2u - a^2cos^2u - 4a^2sin^22u$=0
$$\Rightarrow -\alpha acosu - a\beta sinu + 4a^2cos^22u - 4a^2sin^22u - 4a\gamma cos2u = 0$$
$$\Rightarrow -\alpha acosu - a\beta sinu + 4a^2[cos^22u - sin^22u] - 4a\gamma cos2u = 0$$
$$\Rightarrow -\alpha cosu - \beta sinu + 4acos4u - 4\gamma cos2u = 0$$
$$\Rightarrow \alpha cosu + \beta sinu - 4acos4u + 4\gamma cos2u = 0 \text{ ....(6)}$$
$$\dddot{F}(o) = 0 \Rightarrow (\bar{c} - \bar{r})(\dddot{\bar{r}}) - 3\dot{\bar{r}}\ddot{\bar{r}} = 0$$
$$[(\alpha - acosu), (\beta - asinu), (\gamma - acos2u)]$$
$$(asinu, -acosu, 8asin2u)inu, acosu, -2asin2u)$$
$(-acosu, -asinu, -4acos2u)$=0

$\Rightarrow$

$[(\alpha - acosu)(asinu) + (\beta - asinu)(-acosu) + (\gamma - acos2u)(8asin2u)] - 3[a^2sinucosu - a^2sinucosu + 8a^2sin2ucos2u]$=0
$$\Rightarrow \alpha asinu - \beta acosu + 8\gamma asin2u - 8a^2sin2ucos2u - 24a^2sin2ucos2u = 0$$
$$\Rightarrow \alpha sinu - \beta cosu + 8\gamma sin2u - 16asin4u = 0$$
From (5) and (6)
$$\Rightarrow \frac{\alpha}{-4cosucos2u - 2sinusin2u} = \frac{\beta}{-8asin2ucos4u + 4acos2usin4u}$$
$$= \frac{\gamma}{-asin4ucosu + 4acos4usinu} = \frac{1}{sin^2u + cos^2u}$$
$$\alpha = -4cosucos2u - 2sinusin2u$$
$$\beta = 4acos2usin4u - 8asin2ucos4u$$
$$\gamma = 4acos4usinu - asin4ucosu$$

5. The principle normal to the curve is normal to the locus of centre of curvature at those points then the curvature is stationary.

Solution:

Assume that: The curvature is stationary

To prove: The principle normal is normal to the locus of centre of curvature.

Since the curvature is stationary k=a where a is a constant.

NOTES

$$\Rightarrow \frac{1}{\rho} = a$$

$\Rightarrow \rho = \frac{1}{a}$.....(*)

Diff w.r to s, $\frac{d\rho}{ds} = 0$

(ie) ρ'=0 ...(1)

Since r is the position vector at any point on the curve ρ then we have $r_1 = r + \rho n$ ...(2)

Diff w.r tos, $\frac{dr_1}{ds} = r' + \rho n' + \rho' n$

$$\frac{dr_1}{ds_1}.\frac{ds_1}{ds} = t + \rho(\tau b - kt) + 0$$

$$r'_1.s'_1 = t + \rho\tau b - \rho k t$$

$$= t + \rho\frac{1}{\sigma}b - \rho\frac{1}{\rho}t$$

$$= t + \frac{\rho}{\sigma}b - t$$

$$r'_1.s'_1 = \frac{\rho}{\sigma}b$$

Taking scalar product by n on both sides,

$$r'_1.s'_1.n = \frac{\rho}{\sigma}(b.n)$$

$$t_1.s'_1.n = \frac{\rho}{\sigma}(0)$$

$$t_1.s'_1.n = 0$$

$$t_1.n = 0$$

The principle normal is normal to the locus of centre of curvature.

6. Find the equation of the osculating plane, osculating circle, osculating sphere at the point (1,2,3) on the curve x=2t+1, y=3$t^2 + 2$, z=4t^3+3.

Solution:

(i) Given x=2t+1, y=3$t^2 + 2$, z=4t^3+3

(ie) $\vec{r} = (2t + 1, 3t^2 + 2, 4t^3 + 3)$

$$\dot{\bar{r}} = (2, 6t, 12t^2)$$

$$\ddot{\bar{r}} = (0, 6, 24t)$$

$$\dddot{\bar{r}} = (0, 0, 24)$$

Clearly, t=0 at the point (1,2,3)

At t=0

$$\bar{r} = (1,2,3)$$

$$\dot{\bar{r}} = (2,0,0)$$

$$\ddot{\bar{r}} = (0,6,0)$$

$$\dddot{\bar{r}} = (0,0,24)$$

We know that,

The equation of the osculating plane is $[\bar{R} - \bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}] = 0$

$$[\bar{R} - \bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}] = 0 \Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{vmatrix}$$

$$\Rightarrow (x-1)(0) - (y-2)(0) + (z-3)(12) = 0$$

$$\Rightarrow 0 - 0 + 12z - 36 = 0$$

$$\Rightarrow 12(z-3) = 0$$

$$\Rightarrow z - 3 = 0$$

z=3 .....(1),

which is the required equation of the osculating plane.
(ii) Now, To find the equation of the osculating sphere.
Let (α, β, γ) be the centre of the osculating sphere.
We know that,
The equation of the osculating sphere is $(c-r)^2 = R^2$
Since the osculating sphere has 4 point contact, we have
f(0)=0, $\dot{f}(0) = 0$, $\ddot{f}(0) = 0$, $\dddot{f}(0) = 0$
Here, f(s) =$(c-r)^2 - R^2$

$$f(0) = 0 \Rightarrow (c-r)^2 = R^2$$
$$\dot{f}(0) = 0 \Rightarrow 2(c-r)(-\dot{r}) = 0$$
$$\Rightarrow (c-r)(-\dot{r}) = 0$$
$$(\alpha - 1, \beta - 2, \gamma - 3).(2,0,0) = 0$$
$$\Rightarrow (\alpha - 1).2=0$$
$$\Rightarrow 2\alpha - 2=0$$
$$2\alpha = 2$$
$$\alpha = 1$$
$$\ddot{f}(0) = 0 \Rightarrow (c-r)(\bar{r}) + (\dot{\bar{r}})(-\dot{\bar{r}})=0$$
$$\Rightarrow (c-r)(\bar{r}) - \dot{\bar{r}}^2=0$$
$$(\alpha - 1, \beta - 2, \gamma - 3).(0,6,0) - (2,0,0)^2 = 0$$
$$\Rightarrow 6.(\beta - 2) - (4,0,0) = 0$$
$$\Rightarrow 6\beta - 12 - 4 = 0 \Rightarrow 6\beta - 16 = 0$$
$$\Rightarrow 6\beta = 16 \ \Rightarrow \beta = \frac{16}{6}$$
$$\beta = \frac{8}{3}$$
$$\dddot{f}(0) = 0 \Rightarrow (c-r)(\dddot{\bar{r}}) + (\dot{\bar{r}})(-\dot{\bar{r}}) - 2\dot{\bar{r}}\ddot{\bar{r}}=0$$
$$\Rightarrow (c-r)(\dddot{\bar{r}}) - 3\dot{\bar{r}}\ddot{\bar{r}}=0$$
$$(\alpha - 1, \beta - 2, \gamma - 3).(0,0,24) - 3(2,0,0)(0,6,0) = 0$$
$$\Rightarrow 24(\gamma - 3) - 3(0) = 0$$
$$\Rightarrow 24\gamma - 72 - 0=0$$
$$\Rightarrow 24\gamma = 72$$
$$\Rightarrow \gamma = \frac{72}{24}$$
$$\Rightarrow \gamma = 3$$

To find the radius of the osculating sphere is

$$(c-r)^2 = R^2$$
$$(\alpha - 1, \beta - 2, \gamma - 3)^2 = R^2$$
$$(1-1, \frac{8}{3} - 2, 3-3)^2 = R^2$$
$$(0, \frac{8-6}{3}, 0)^2 = R^2$$
$$(\frac{2}{3}) = R^2$$
$$R = \frac{2}{3}$$

The equation of the osculating sphere is $(c-r)^2 = R^2$

$$(\alpha - x, \beta - y, \gamma - z)^2 = R^2$$
$$(1-x, \frac{8}{3} - y, 3-z)^2 = \frac{4}{9}$$

$$(1-x)^2+(\frac{8}{3}-y)^2+(3-z)^2=\frac{4}{9}$$
$$1+x^2-2x+\frac{64}{9}+y^2-\frac{16}{3}y+9+z^2-6z=\frac{4}{9}$$
$$x^2+y^2+z^2-2x-\frac{16}{3}y-6z+10+\frac{64}{9}-\frac{4}{9}=0$$
$$x^2+y^2+z^2-2x-\frac{16}{3}y-6z+10+\frac{60}{9}=0$$
$$x^2+y^2+z^2-2x-\frac{16}{3}y-6z+\frac{60+90}{9}=0$$
$$x^2+y^2+z^2-2x-\frac{16}{3}y-6z+\frac{150}{9}=0$$
$$x^2+y^2+z^2-2x-\frac{16}{3}y-6z+\frac{50}{3}=0$$
$$3x^2+3y^2+3z^2-6x-16y-18z+50=0 \text{ .....(2)}$$

(iii) Osculating circle is the intersection of the osculating plane and the osculating sphere.

Sub (1) in (2)

$3x^2+3y^2+3(3)^2-6x-16y-18(3)+50=0$

$3x^2+3y^2+27-6x-16y-54+50=0$

$3x^2+3y^2-6x-16y+23=0$

which is the required equation of the osculating circle.

7. Show that the radius of spherical curvature of the helix x=acosu. y=asinu, z=aucosα is equal to the radius of circular curvature of the helix.

Solution:

Given $\vec{r}=(acosu, asinu, aucos\alpha)$

$$\dot{\vec{r}}=(-asinu, acosu, acos\alpha)$$
$$\ddot{\vec{r}}=(-acosu, -asinu, 0)$$
$$\dot{\vec{r}}\times\ddot{\vec{r}}=\begin{vmatrix}\bar{t} & \bar{n} & \bar{b}\\ -asinu & acosu & acos\alpha\\ -acosu & -asinu & 0\end{vmatrix}$$

$=\bar{t}(0+a^2sinucos\alpha)+\bar{b}(a^2sin^2u+a^2cos^2u)-\bar{n}(0+a^2cos\alpha cosu)$

$=a^2sinucos\alpha\bar{t}+a^2\bar{b}-a^2cos\alpha cosu\bar{n}$

$$|\dot{\vec{r}}\times\ddot{\vec{r}}|=\sqrt{(a^2sinucos\alpha)^2+(a^2)^2+(a^2cos\alpha cosu)^2}$$

$=\sqrt{a^4sin^2ucos^2\alpha+a^4+a^4cos^2\alpha cos^2u}$

$=a^2\sqrt{cos^2\alpha(sin^2u+cos^2u)+1}$

$|\dot{\vec{r}}\times\ddot{\vec{r}}|=a^2\sqrt{cos^2\alpha+1}$ ....(1)

$$|\dot{\vec{r}}|=\sqrt{(-asinu)^2+(acosu)^2+(acos\alpha)^2}$$
$$=\sqrt{a^2sin^2u+a^2cos^2u+a^2cos^2\alpha}$$
$$=a\sqrt{(sin^2u+cos^2u)+cos^2\alpha}$$
$$|\dot{\vec{r}}|=a\sqrt{1+cos^2\alpha}$$
$$|\dot{\vec{r}}|^3=a^3(1+cos^2\alpha)^{\frac{3}{2}}$$

$=a^3(1+cos^2\alpha)\sqrt{1+cos^2\alpha}$ ....(2)

We know that,

k $=\frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}=\frac{a^2\sqrt{cos^2\alpha+1}}{a^3(1+cos^2\alpha)\sqrt{1+cos^2\alpha}}=\frac{1}{a(1+cos^2\alpha)}$ =b(say)

we know that,

$R^2=\rho^2+\rho'^2\sigma^2$ .....(4)

k $=\frac{1}{\rho}=b$

$$\Rightarrow \frac{1}{\rho} = b$$
$$\Rightarrow \rho = \frac{1}{b}$$

ρ'=0

(4) $\Rightarrow R^2 = \rho^2$

R=ρ

The radius of the spherical curvature of the helix is equal to the radius of the circular curvature of the helix.

Tangent surface:

A surface of a curve c is called a tangent surface if the surface generated bt tangent to the curve c.

8. Find the equation of the tangent surface.

Solution:

Let $\bar{r} = \bar{r}(s)$ be the equation of the curve c.

Let p(r) be any point on the curve c and Q(R) be the neighbouring point corresponding to p on the tangent surface.

Now $\bar{P}Q$ is the tangent to c.

$$\Rightarrow \bar{P}Q = \lambda\bar{t}$$
$$\Rightarrow \bar{O}Q - \bar{O}P = \lambda\bar{t}$$
$$\Rightarrow R - \bar{r} = \lambda\bar{t}$$
$$\Rightarrow R = \lambda\bar{t} + \bar{r}$$

(ie) R=$\bar{r}(s) + \lambda\bar{t}$

This is the equation of the tangent surface.

9. Find the equation of the tangent surface to the curve $\bar{r} = (u, u^2, u^3)$.

Solution:

We know that,

The equation of the tangent surface is R=$\bar{r}(s) + \lambda\bar{t}$

R=$\bar{r}(s) + \lambda\dot{\bar{r}}$

=$(u, u^2, u^3) + \lambda(1, 2u, 3u^2)$

=$(u + \lambda, u^2 + 2u\lambda, u^3 + 3u^2\lambda)$, which is the equation of the tangent surface.

Let c and c_1 be the two curves. The curve c_1 is called involute of c if the tangent to c is normal to c_1.

If c_1 is the involute of c then c is called evolute of c_1.

Theorem 2.5 *Find the equation of the involute of the curve c.*

Proof:

Let the equation of the given curve c be $\bar{r} = \bar{r}(s)$ and the equation of the involute of c_1 be $\bar{r}_1 = \bar{r}_1(s)$.

Let p(r) and $p_1(r_1)$ be the corresponding points on c and c_1 respectively.

Let 'o' be the origin then $\bar{O}P_1 = \bar{O}P + \bar{P}P_1$

$r_1 = \bar{r} + \lambda t$ .....(1)

Diff (1) w.r to s_1,

$$\frac{dr_1}{ds_1} = \frac{dr}{ds_1} + \lambda\frac{dt}{ds_1} + t\frac{d\lambda}{ds_1}$$
$$r'_1 = \frac{dr}{ds}.\frac{ds}{ds_1} + \lambda\frac{dt}{ds}.\frac{ds}{ds_1} + t\frac{d\lambda}{ds}.\frac{ds}{ds_1}$$
$$r'_1 = r'.s' + \lambda.t'.s' + t.\lambda'.s'$$

$$r'_1 = t.s' + \lambda.\frac{dt}{ds}.s' + t.\lambda'.s'$$
$$r'_1 = t.s' + \lambda.k\bar{n}.s' + t.\lambda'.s'$$
$$t_1 = t.\frac{ds}{ds_1} + \lambda.k\bar{n}.\frac{ds}{ds_1} + t.\lambda'.\frac{ds}{ds_1}$$
$$t_1 = \frac{ds}{ds_1}(t + \lambda.k\bar{n} + t.\lambda') \text{ .....(2)}$$

Now $t.t_1 = 0$

Taking scalar product on both sides by t

(2) $\Rightarrow t_1.t = \frac{ds}{ds_1}(t + \lambda.k\bar{n} + t.\lambda').t$

$0=\frac{ds}{ds_1}[(t.t) + \lambda.k(\bar{n}.t) + (t.t)\lambda')]$

$0=\frac{ds}{ds_1}(1 + \lambda')$

$0=ds(1 + \lambda') \Rightarrow ds(1 + \lambda') = 0$

$$\Rightarrow ds + ds\lambda' = 0$$
$$\Rightarrow ds + \frac{d\lambda}{ds}ds = 0$$
$$\Rightarrow ds + d\lambda = 0$$

Integrating

$s + \lambda$=a where a is a constant.

$\Rightarrow \lambda = a - s$.....(3)

Sub (3) in (1),

$$r_1 = \bar{r} + \lambda t$$

$r_1 = \bar{r} + (a - s)t$ ....(4)

Sub λ, λ' value in (2)

(2) $\Rightarrow \bar{t}_1 = (t + \lambda kn + t\lambda').\frac{ds}{ds_1}$

$$= (t + (a - s)kn + t(-1)).\frac{ds}{ds_1}$$
$$= (t + (a - s)kn - t).\frac{ds}{ds_1}$$
$$=(a - s)kn.\frac{ds}{ds_1}$$
$$\Rightarrow \bar{t}_1.\frac{ds_1}{ds} = (a - s)kn$$

Now, taking modulus, we get

$$|\bar{t}_1.\frac{ds_1}{ds}| = \sqrt{(a - s)^2k^2}$$
$$\sqrt{(\frac{ds_1}{ds})^2} = \sqrt{(a - s)^2k^2}$$
$$(\frac{ds_1}{ds}) = (a - s)k$$

The equation of the involute is $r_1 = r + (a - s)t$ and s can be obtain in terms of s_1 from the eqn (5).

Theorem 2.6 *Find the equation of the evolute of the curve c.*

Proof:

Let the equation of the curve be $\vec{r} = \vec{r}(s)$ and c' be its evolute. Then c is the involute of c'.

Let p(r) be any point on c and Q(R) be the corresponding point on c'.

Then PQ lies on the normal plane at p to c.

$\Rightarrow PQ, \vec{n}, \vec{b}$ are coplanar.

$\Rightarrow PQ = \lambda\vec{n} + \mu\vec{b}$ where λ and μ are scalars.

$$\Rightarrow OQ - OP = \lambda\vec{n} + \mu\vec{b}$$

$$\Rightarrow OQ = OP + \lambda\vec{n} + \mu\vec{b}$$

$\Rightarrow \bar{R} = \bar{r} + \lambda\vec{n} + \mu\vec{b}$ .....(1)

Diff (1) w.r to s,

$$\frac{d\bar{R}}{ds} = \overline{r'} + \lambda\overline{n'} + \lambda' n + \mu b' + \mu'\bar{b}$$

$=\bar{t} + \lambda(\tau b + kt) + \lambda' n + \mu(-\tau n) + \mu'\bar{b}$

$=\bar{t}(1 - k\lambda) + (\lambda' - \mu\tau)\bar{n} + (\lambda\tau + \mu')\bar{b}$ ....(2)

Here R' is parallel to $\bar{P}Q = \lambda\bar{n} + \mu\bar{b}$

$\Rightarrow \bar{t}(1 - k\lambda) + (\lambda' - \mu\tau)\bar{n} + (\lambda\tau + \mu')\bar{b} = \lambda\bar{n} + \mu\bar{b}$

Equating the co-efficient of $\bar{t}, \bar{n}, \bar{b}$ on both sides,

$\Rightarrow 1 - k\lambda = 0$ ....(3)

$\Rightarrow \lambda' - \mu\tau = \lambda$ ...(4)

$\Rightarrow \lambda\tau + \mu' = \mu$ ....(5)

(3) $\Rightarrow \lambda = \frac{1}{k} = \rho$

(4) $\Rightarrow \frac{\lambda' - \mu\tau}{\lambda} = 1$ and

(5) $\Rightarrow \frac{\lambda\tau + \mu'}{\mu} = 1$

$$\Rightarrow \frac{\lambda' - \mu\tau}{\lambda} = \frac{\lambda\tau + \mu'}{\mu}$$

$$\Rightarrow \mu(\lambda' - \mu\tau) = \lambda(\lambda\tau + \mu')$$

$$\Rightarrow \mu\lambda' - \mu^2\tau = \lambda^2\tau + \lambda\mu'$$

$$\Rightarrow \lambda^2\tau + \lambda\mu' - \mu\lambda' + \mu^2\tau = 0$$

$$\Rightarrow (\lambda^2 + \mu^2)\tau = (\mu\lambda' - \lambda\mu')$$

$$\tau = \frac{\mu\lambda' - \lambda\mu'}{\lambda^2 + \mu^2}$$

$$\tau = \frac{d}{ds}[tan^{-1}(\frac{\lambda}{\mu})]$$

$$\Rightarrow tan^{-1}(\frac{\lambda}{\mu}) = \int \tau . ds + c$$

$$\Rightarrow (\frac{\lambda}{\mu}) = tan[\int \tau . ds + c]$$

$$\Rightarrow \lambda = \mu tan[\int \tau . ds + c]$$

$$\Rightarrow \mu = \lambda cot[\int \tau . ds + c]$$

$$\Rightarrow \mu = \rho cot[\int \tau . ds + c]$$

Sub μ and λ in (1), we get

$\Rightarrow \bar{R} = \bar{r} + \rho\vec{n} + \rho cot[\int \tau . ds + c]\vec{b}$

Theorem 2.7 *Find the equation of the curvature and torsion of the involute of the curve c.*

Proof:

Let c_1 be the involute of c.

The unit tangent vector along c_1 is given $t_1 = \bar{n}_1 = n$ ....(1)

Diff (1) w.r to s_1, $\frac{dt_1}{ds_1} = \frac{dn}{ds_1}$

$$\Rightarrow t'_1 = \frac{dn}{ds}.\frac{ds}{ds_1} = (\tau b - kt)\frac{1}{(a-s)k}$$

$$k_1 n_1 = \frac{\tau b - kt}{(a-s)k}$$

$$k_1 n_1 = \frac{\tau b}{(a-s)k} - \frac{kt}{(a-s)k} \quad .....(2)$$

Taking scalar product on both sides,

$$k_1^2 = \frac{\tau^2}{(a-s)^2k^2} + \frac{k^2}{(a-s)^2k^2}$$

$$= \frac{\tau^2+k^2}{(a-s)^2k^2} \quad ....(3)$$

(2) $\Rightarrow \frac{dt_1}{ds_1} = \frac{\tau b - kt}{(a-s)k}$

$$\Rightarrow \frac{d}{ds_1}\left(\frac{dr_1}{ds_1}\right) = \frac{\tau b - kt}{(a-s)k}$$

$$\frac{d^3}{ds_1^3}(r_1) = \frac{d}{ds_1}\left[\frac{\tau b - kt}{(a-s)k}\right].\frac{d}{ds}\left(\frac{\tau b - kt}{(a-s)k}\right).\frac{ds}{ds_1}$$

$$= \frac{(a-s)k(\tau' b + \tau b' - k' t - k t') - (\tau b - kt)[(a-s)k' + k(-1)]}{(a-s)^2k^2}.\frac{1}{(a-s)k}$$

$$= \frac{(a-s)k[\tau' b + \tau(-\tau n) - k' t - k(k\bar{n})] - (\tau b - kt)[(a-s)k' - k]}{(a-s)^3k^3}$$

$$= \frac{-(a-s)kk' + (a-s)kk' - k^2]\bar{t} + [(a-s)k(-\tau^2 - k)]\bar{n} + [(a-s)k\tau' - \tau(a-s)k' + \tau k]\bar{b}}{(a-s)^3k^3}$$

$$\vec{r}''' = \frac{-k^2 t - [(a-s)(\tau^2+k^2)]\bar{n} + [((a-s)k\tau' - \tau k') + \tau k]\bar{b}}{(a-s)^3k^3}$$

$$k_1^2\tau_1 = \left[\frac{dr_1}{ds_1}, \frac{d^2r_1}{ds_1^2}, \frac{d^3r_1}{ds_1^3}\right]$$

$$= \left[\bar{n}, \frac{\tau\bar{b} - k\bar{t}}{(a-s)k}, \frac{-k^2 t - [(a-s)(\tau^2+k^2)]\bar{n} + [((a-s)k\tau' - \tau k') + \tau k]\bar{b}}{(a-s)^3k^3}\right]$$

$$= \begin{vmatrix} 0 & 1 & 0 \\ \frac{-k}{(a-s)k} & 0 & \frac{\tau}{(a-s)k} \\ \frac{-k^2}{(a-s)^3k^3} & \frac{-(a-s)(\tau^2+k^2)}{(a-s)^3k^3} & \frac{((a-s)k\tau' - \tau k') + \tau k}{(a-s)^3k^3} \end{vmatrix}$$

$$= -1\left\{\left\{\frac{-k}{(a-s)k}\left[\frac{((a-s)k\tau' - \tau k') + \tau k}{(a-s)^3k^3}\right]\right\} - \left\{\frac{\tau}{(a-s)k}.\frac{-k^2}{(a-s)^3k^3}\right\}\right\}$$

$$= \frac{k(a-s)(k\tau' - k'\tau) + k^2\tau - \tau k^2}{(a-s)^4k^4}$$

$$k_1^2\tau_1 = \frac{k\tau' - k'\tau}{(a-s)^3k^3}$$

$$\tau_1 = \frac{k\tau' - k'\tau}{(a-s)^3k^3}.\frac{1}{k_1^2}$$

$$= \frac{k\tau' - k'\tau}{(a-s)^3k^3}.\frac{(a-s)^2k^2}{\tau^2+k^2}$$

$$\tau_1 = \frac{k\tau' - k'\tau}{(a-s)k(\tau^2+k^2)}$$

10. Show that the torsion of the involute of a given curve is equal to $\frac{\rho(\sigma\rho' - \sigma'\rho)}{(\rho^2+r^2)(a-s)}$

Solution:

The torsion of the involute of c is given by

$$\tau_1 = \frac{k\tau' - k'\tau}{(a-s)k(\tau^2+k^2)} \quad .....(1)$$

Put k=$\frac{1}{\rho}$ and $\tau = \frac{1}{\sigma}$
k'=$\frac{-\rho\prime}{\rho^2}$ and $\tau = \frac{-\sigma\prime}{\sigma^2}$
(1) $\Rightarrow \tau_1 = \frac{\frac{1}{\rho}(\frac{-\sigma\prime}{\sigma^2})-(\frac{-\rho\prime}{\rho^2})(\frac{1}{\sigma})}{(a-s)(\frac{1}{\rho})(\frac{1}{\rho^2}+\frac{1}{\sigma^2})}$ =$\frac{-\rho\sigma\prime+\rho\prime\sigma}{\rho^2\sigma^2} / \frac{(a-s)(\sigma^2+\rho^2}{\rho(\rho^2\sigma^2)}$

$\tau_1 = \frac{\rho(\sigma\rho\prime-\sigma\prime\rho)}{(\rho^2+r^2)(a-s)}$

Theorem 2.8 *Prove that the locus of the centre of curvature of a given curve is an evolute only when the curve is a plane curve.*

Proof:

The position vector of a current point in the evolute is given by
$\bar{R} = \bar{r} + \rho\bar{n} + \rho cot(\int \tau ds + c)\bar{b}$.....(1)
The locus of centre of curvature is given by $\bar{c} = \bar{r} + \rho\bar{n}$ ....(2)
Eqn (1) and (2) represent the same curve.

$$\rho cot(\int \tau ds + c)\bar{b} = 0$$
$$\Rightarrow \rho = 0 (or) cot(\int \tau ds + c)\bar{b} = 0$$
$$\Rightarrow \rho \neq 0 (or) cot(\int \tau ds + c)\bar{b} = 0$$
$$\Rightarrow \frac{cos(\int \tau ds + c)\bar{b}}{sin(\int \tau ds + c)\bar{b}} = 0$$
$$\Rightarrow cos(\int \tau ds + c)\bar{b} = 0$$
$$\Rightarrow cos(\int \tau ds + c) = 0$$
$$\Rightarrow \int \tau ds + c = cos^{-1}(0)$$
$$\Rightarrow \int \tau ds + c = multiple of \frac{\Pi}{2}$$
$$\Rightarrow \frac{d}{ds}\int \tau ds = 0$$
$$\tau = 0$$

The curve is a plane curve.

Theorem 2.9 *Show that the involute of a circular helix are plane curves.*

Proof:

The torsion of the involute of a curve $\vec{r} = \vec{r}(s)$ is given by
$\tau_1 = \frac{k\tau\prime - k\prime\tau}{(a-s)k(\tau^2+k^2)}$ ....(1)
Since the given curve is a circular helix, we have,
$\frac{k}{\tau}$=b where $\bar{b}$ is a circular helix.

$$\Rightarrow k = \tau b$$
$$\Rightarrow k' = \tau' b$$

0$\Rightarrow \tau_1 = \frac{\tau b\tau\prime - \tau\prime b\tau}{(a-s)k(\tau^2+k^2)}$

$$\Rightarrow \tau_1 = 0$$

The involute of a circular helix is a plane curves.

Theorem 2.10 *If the position vector $\bar{r}$ of a current point on a curve is a function of any parameter u and dots denote differentiate with respect to u* *then prove that* $\dot{\vec{r}} = \dot{s}\vec{t}$, $\ddot{\vec{r}} = \ddot{s}\vec{t} + k\dot{s}^2\vec{n}$, $\dddot{\vec{r}} = (\dddot{s} - k^2\dot{s}^3)\vec{t} + \dot{s}(3k\ddot{s})\vec{n} + (k\tau\dot{s}^3)\vec{b}$ *and hence deduce that*

$$\bar{b} = \frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{k\dot{s}^3}, \bar{n} = \frac{\dot{s}\ddot{\bar{r}} - \ddot{s}\dot{\bar{r}}}{k\dot{s}^3}$$

$$k^2 = \frac{\ddot{\bar{r}}^2 - \ddot{s}^2}{\dot{s}^4}, \tau = \frac{[\dot{\bar{r}}\ddot{\bar{r}}\dddot{\bar{r}}]}{\dot{k}\dot{s}^6}$$

Proof:

$$\dot{\bar{r}} = \frac{d\bar{r}}{du} = \frac{d\bar{r}}{ds}.\frac{ds}{du}$$

$$\dot{\bar{r}} = \overline{r'}.\dot{s}$$

$\dot{\bar{r}} = \dot{s}\bar{t}$ .....(1)

Diff (1) w.r to u, $\ddot{\bar{r}} = \ddot{s}\bar{t} + (\dot{s}^2)\overline{t'}$

$\ddot{\bar{r}} = \ddot{s}\bar{t} + k\dot{s}^2\bar{n}$ ......(2)

Diff (2) w.r to u,

$$\dddot{\bar{r}} = \dddot{s}\bar{t} + \ddot{s}\dot{s}\overline{t'} + (k\dot{s}^3 + 2k\dot{s}\ddot{s})\bar{n} + k\dot{s}^3\overline{b'}$$

$=\dddot{s}\bar{t} + \ddot{s}\dot{s}k\bar{n} + (k\dot{s}^3 + 2k\dot{s}\ddot{s})\bar{n} + k\dot{s}^3(\tau\bar{b} - k\bar{t})$

$=(\dddot{s} - k^2\dot{s}^3)\bar{t} + \dot{s}(3k\ddot{s} + k\dot{s})\bar{n} + (k\tau\dot{s}^3)\bar{b}$ ......(3)

Taking cross product (1) and (2)

$$\dot{\bar{r}} \times \ddot{\bar{r}} = (\dot{s}\ddot{s}\bar{t} \times t) + (k\dot{s}^3 t \times n)$$

$\dot{\bar{r}} \times \ddot{\bar{r}} = k\dot{s}^3 b$ .....(4)

$$\Rightarrow \bar{b} = \frac{\dot{\bar{r}} \times \ddot{\bar{r}}}{k\dot{s}^3}$$

$(2) \times \dot{s} - (1) \times \ddot{s}$

$$\Rightarrow \ddot{\bar{r}}\dot{s} - \ddot{s}\dot{\bar{r}} = k\dot{s}^3\bar{n}$$

$\Rightarrow \bar{n} = \frac{\ddot{\bar{r}}\dot{s} - \ddot{s}\dot{\bar{r}}}{k\dot{s}^3}$ .....(5)

Taking scalar product of (2)

$$\Rightarrow \ddot{\bar{r}}^2 = \ddot{s}^2 + k^2\dot{s}^2$$

$$\Rightarrow k^2 = \frac{\ddot{\bar{r}}^2 - \ddot{s}^2}{\dot{s}^4}$$

Taking scalar product of (3) and (4)

$$\Rightarrow (\dot{\bar{r}} \times \ddot{\bar{r}}) \times \dddot{\bar{r}} = k^2\tau\dot{s}^6$$

$$\Rightarrow [\dot{\bar{r}}\ddot{\bar{r}}\dddot{\bar{r}}] = k^2\tau\dot{s}^6$$

$\tau = \frac{[\dot{\bar{r}}\ddot{\bar{r}}\dddot{\bar{r}}]}{k^2\dot{s}^6}$

3.4 Check your progress

1.Define osculating sphere.

2.Define centre of spherical curvature

3. Define osculating circle

4. Define involute and evolute of the curve.

3.5 Summary

The surface F(f(u),g(u),h(u)) =0. If u_0 is a zero of F(u)=0 then F(u) can be expressed by the Taylor's theorem,

F(u) = $\varepsilon F'(u_0) + \frac{\varepsilon^2}{2!}F''(u_0) + \ldots + \frac{\varepsilon^n}{n!}F^{(n)}(u_0) + O(\varepsilon^{n+1})$, where $\varepsilon = u - u_0$

The center of the osculating sphere is $\vec{c} = \vec{r} + \rho\vec{n} + \rho'\sigma\vec{b}$

R=$\sqrt{\rho^2 + \rho'^2\sigma^2}$ is the radius of the osculating sphere.

The radius of curvature of the locus of the centre of curvature of a curve is given

$$[(\frac{\rho^2\sigma}{R^3}\frac{d}{ds}(\frac{\sigma\rho'}{\rho})-\frac{1}{R})^2+\frac{\rho'\sigma^4}{\rho^2R^4}]^{-\frac{1}{2}}$$

The equation of the osculating plane is $[\bar{R}-\bar{r},\dot{\bar{r}},\ddot{\bar{r}}]=0$

Osculating circle is the intersection of the osculating plane and the osculating sphere.

3.6 Keywords

Osculating sphere: The osculating sphere at a point on a curve is the sphere which has four points contact with the curve at p.

Osculating circle: Osculating circle at any point p on a curve is a circle which has three points contact with the curve at the point p. It is also known as circle of curvature.

Tangent surface: A surface of a curve c is called a tangent surface if the surface generated bt tangent to the curve c.

The torsion of the involute : $\tau_1=\frac{k\tau\prime-k\prime\tau}{(a-s)k(\tau^2+k^2)}$

3.7 Self Assessment Questions and Exercises

1.Find the equation of the osculating plane of the curve given by r=(asinu+bcosu, acosu+bsinu, csin2u). Find also the radius of spherical curvature at any point.

2. Find the osculating sphere and osculating circles at the point (1,2,3) on the curve r=(2u+1, $3u^2+2, 4u^3$+3).

3. Find the osculating sphere at any point of a circular helix.

4. Prove that r=(a cos^2u, a cosu sinu, a sinu) is a spherical curve.

5. Show that the tangent to the locus C_1 of the centres of curvature lies in the normal plane of the given curve C. If θ is the angle between the tangent to C_1 and the principal normal of C, prove that tan θ=$\frac{\rho}{\rho'\sigma}$

6. If s is the arc length of the locus of centres of curvature, show that $\frac{ds_1}{ds}$=$\frac{\sqrt{\kappa^2\tau^2+\kappa'^2}}{\kappa^2}$

7. If C is a curve of constant curvature κ, show that the locus C_1 of centres of curvature is also a curve of constant curvature $\kappa_1$such that κ_1=κ and that its torsion is given by τ_1=$\frac{\kappa^2}{\tau}$.

8. If R is the radius of spherical curvature for any point P(x,y,z) on the curve, prove that $(x''')^2+(y''')^2+(z''')^2=\frac{1}{\rho^4}+\frac{R^2}{\rho^4\sigma^2}$.

9. Show that the torsion at corresponding points P and P_1 of two Bertrand curves have the same sign and that their product is constant. If C, C_1 are their centres of curvature, prove that the cross ratio (PCP_1C_1) is the same for all corresponding pairs of points.

10. The locus of a point whose position vector is the binormal b of a curve ϒ is called the spherical indicatrix of the binormal to ϒ. Prove that its curvature κ_2 and torsion τ_2 are given by ${\kappa_2}^2$=$\frac{\kappa_2+\tau_2}{\tau_2}$, $\tau_2=\frac{\tau\kappa'-\kappa\tau'}{\tau(\kappa_2+\tau_2)}$.

NOTES

3.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

UNIT-IV INTRINSIC EQUATIONS

NOTES

Structure

4.1 Introduction

In this chapter we discuss about the intrinsic equations of space curves and establish the fundamental theorem of space curves which states that if curvature and torsion are the given continuous functions of a real variable s,then they determine the space curve uniquely.

4.2 Objectives

4.3 Intrinsic Equations

The equation k=f(s), $\tau = g(s)$ are called intrinsic equation.

Theorem 2.11 ***Fundamental existence theorem for space curve***

If k(s) and τ(s) are continuous function of the real variable s≥0 then there exists a space curve for which k is a curvature and τ is a torsion and s is an arc length measured from suitable base point.

Proof:

Given k(s) and τ(s) are continuous function of the real variable s≥0.

To Prove: There exists a space curve for which k is a curvature and τ is a torsion and s is an arc length measured from suitable base point.

Consider the differential equation.

$\frac{d\alpha}{ds} = k\beta, \frac{d\beta}{ds} = \tau\gamma - k\alpha, \frac{d\gamma}{ds} = -\tau\beta$ .....(1)

Then (1) have unique set of solution for given set of values α, β, γ at s=0.

In particular there is a unique set $(\alpha_1, \beta_1, \gamma_1)$ which assume values (1,0,0) when s=0.

Similarly, there is a unique set $(\alpha_2, \beta_2, \gamma_2)$ which assumes initial values (0,1,0) and unique set $(\alpha_3, \beta_3, \gamma_3)$ which assumes initial values (0,0,1) at s=0.

Now, we prove that for all values of s, $\alpha_1^2 + \beta_1^2 + \gamma_1^2$=1.

$$\frac{d}{ds}(\alpha_1^2 + \beta_1^2 + \gamma_1^2 = 2\alpha_1 \frac{d\alpha_1}{ds} + 2\beta_1 \frac{d\beta_1}{ds} + 2\gamma_1 \frac{d\gamma_1}{ds}$$

$$=2\alpha_1(k\beta_1) + 2\beta_1(\tau\gamma_1 - k\alpha_1) + 2\gamma_1(-\tau\beta_1)$$

$$=0$$

Integrating we get,

$\alpha_1^2 + \beta_1^2 + \gamma_1^2 = \lambda_1$(say) .....(A)

(A) $\Rightarrow 1 + 0 + 0 = \lambda_1$

$$\lambda_1 = 1$$

NOTES

(A) $\Rightarrow \alpha_1^2+\beta_1^2+\gamma_1^2$=1.
Similarly we get
$\alpha_2^2+\beta_2^2+\gamma_2^2$=1
$\alpha_3^2+\beta_3^2+\gamma_3^2$=1 ....(2)
Consider the point

$$\frac{d}{ds}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)=\alpha_1\frac{d\alpha_2}{ds}+\beta_1\frac{d\beta_2}{ds}+\gamma_1\frac{d\gamma_2}{ds}$$

$=\alpha_1(k\beta_2)+\alpha_2(k\beta_1)+\beta_1(\tau\gamma_2-k\alpha_2)+\beta_2(\tau\gamma_1-k\alpha_1)+\gamma_1(-\tau\beta_2)+\gamma_2(-\tau\beta_1)$
=0
Integrating we get,
$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=\mu_1$(say)
At s=0, $(\alpha_1,\beta_1,\gamma_1)$=(1,0,0) and
$(\alpha_2,\beta_2,\gamma_2)$=(0,1,0)
(ie) $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2$=0
$\alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3$=0
$\alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1$=0 ......(3)

Take A= $\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}$

$$AA^T=\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{vmatrix}\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=I$$

A is a orthogonal matrix
(ie) AA^T=I

$$A^T=A^{-1}$$

Post multiply by A,
A^TA=I
We have,

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1$$
$$\beta_1^2+\beta_2^2+\beta_3^2=1$$
$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1$$

And

$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0$$
$$\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3=0$$
$$\beta_1\gamma_1+\beta_2\gamma_1+\beta_1\gamma_1=0$$

There exists 3 mutually orthogonal unit vectors $\bar{t}=(\alpha_1,\alpha_2,\alpha_3)$, $\bar{n}=(\beta_1,\beta_2,\beta_3)$ $\bar{b}=(\gamma_1,\gamma_2,\gamma_3)$ defined for each value of s.
Now define r=r(s)=$\int_0^s tds$ .....(4)
Then $|r'|$=$|\bar{t}|=1$

$$t'=(\frac{d\alpha_1}{ds},\frac{d\alpha_2}{ds},\frac{d\alpha_3}{ds})$$
$$=(k\beta_1,k\beta_2,k\beta_3)$$
$$=k(\beta_1,\beta_2,\beta_3)$$
$$\overline{t'}=k\bar{n}$$

k is a curvature of (4)
Also, $\bar{b}=(\gamma_1,\gamma_2,\gamma_3)$

NOTES

$$b' = (\frac{d\gamma_1}{ds}, \frac{d\gamma_2}{ds}, \frac{d\gamma_3}{ds})$$
$$=(-\tau\beta_1, -\tau\beta_2, -\tau\beta_3)$$
$$=-\tau(\beta_1, \beta_2, \beta_3)$$
$$b' = \frac{db}{ds} = -\tau\bar{n}$$

τ is a torsion of eqn(4).

Hence there exists a space curve given by (4) where $\bar{t}, \bar{b}, \bar{n}$ are unit vectors along tangent, normal and binormal respectively.

Theorem 2.12 ***Uniqueness theorem***

If the curvature and torsion of c have the same values as the curvature and torsion at the corresponding point of c_1 thenc_1 and c are congruent.

Proof:

Let c and c_1 be two curves defined interms of respective arc length such that the points with the same values of s (arc length) corresponds.

Also the curvature and torsion are same for the same value of s.

Let c_1 be moved so that the two points c and c_1 corresponding to s=0 co-incide.

(ie) $(\bar{t}, \bar{n}, \bar{b})$=$(\bar{t}_1, \bar{n}_1, \bar{b}_1)$

Then $\frac{d}{ds}(t.t_1) = t'.t_1 + t'_1.t$

$=k\bar{n}.t_1 + k_1 n_1 t$

$\frac{d}{ds}(t.t_1) = k(\bar{n}.t_1) + k_1(n_1.t)$ .....(1)

Similarly $\frac{d}{ds}(n.n_1) = (\overline{n}'_1.n) + (n_1.n')$

$\frac{d}{ds}(n.n_1) = (\tau b - kt)\bar{n}_1 + (\tau_1 b_1 - k_1 t_1)\bar{n}_1 n$ ....(2)

$\frac{d}{ds}(b.b_1) = b_1(-\tau n) + (-\tau_1 n_1)b$ ....(3)

(1)+(2)+(3) $\Rightarrow \frac{d}{ds}(t.t_1) + \frac{d}{ds}(n.n_1) + \frac{d}{ds}(b.b_1)$=0

$\Rightarrow \frac{d}{ds}(t.t_1 + n.n_1 + b.b_1)$=0

Integrating,

$t.t_1 + n.n_1 + b.b_1$=a

(ie)$t.t_1 + n.n_1 + b.b_1$=3

Since the sum of the cosines is equal to 3 only, when each angle is 0 (s=0)

$cos\alpha + cos\beta + cos\gamma$=0

Now, $cos\alpha = t.t_1 = 1$

$$\Rightarrow \alpha = cos^{-1}(1)$$
$$\Rightarrow \alpha = 0$$

Similarly $\beta = 0, \gamma = 0$

We know that ,

$$t = t_1, n = n_1, b = b_1$$
$$\Rightarrow \frac{dr}{ds} = \frac{dr_1}{ds}$$
$$\Rightarrow \frac{dr}{ds} - \frac{dr_1}{ds} = 0$$
$$\Rightarrow \frac{d}{ds}(r - r_1) = 0$$
$$\Rightarrow d(r - r_1) = 0$$

Integrating,

$r - r_1$=b

NOTES

At s=0 ⇒ $r = r_1$
b=0
$r - r_1$=0
Hence $r - r_1$ for corresponding points.
Hence the two curves are identical and k and τ of c_1 and c are same.

4.4 Check your progress

- Define intrinsic equation
- State fundamental theorem for space curves
- State uniqueness theorem on space curves

4.5 Summary

- **Intrinsic equation**
  The equation k=f(s), $\tau = g(s)$ are called intrinsic equation.
- ***Fundamental existence theorem for space curve***
  If k(s) and τ(s) are continuous function of the real variable s≥0 then there exists a space curve for which k is a curvature and τ is a torsion and s is an arc length measured from suitable base point.

- ***Uniqueness theorem***
  If the curvature and torsion of c have the same values as the curvature and torsion at the corresponding point of c_1 thenc_1 and c are congruent.

4.6 Keywords

- **Intrinsic equation**
  The equation k=f(s), $\tau = g(s)$ are called intrinsic equation.
- ***Fundamental existence theorem for space curve***
  If k(s) and τ(s) are continuous function of the real variable s≥0 then there exists a space curve for which k is a curvature and τ is a torsion and s is an arc length measured from suitable base point.

- ***Uniqueness theorem***
  If the curvature and torsion of c have the same values as the curvature and torsion at the corresponding point of c_1 thenc_1 and c are congruent.

4.7 Self Assessment Questions and Exercises

1. Prove that the corresponding points of the spherical indicatrix of the tangent to C and the indicatrix of the binormal to C have parallel tangent lines.

2. Show that the spherical indicatrix of a curve is a circle if and only if the curve is a helix.

3. Prove that for any curve lying on the surface of a sphere, $\frac{d}{dp}(\sigma\rho') + \frac{\rho}{\sigma}$=0.

4. Prove that corresponding points on the spherical indicatrix of the tangent to ϒ and on the indicatrix of the binormal to ϒ have parallel tangent lines.

5. Find the equation of a tangent surface to the curve r=(u, u^2, u^3).
6. Find the Bertrand associate of a circle in a plane.
7. Prove that the position vector of a current point r=r(s) on a curve satisfies the differential equation $\frac{d}{ds}\left[\sigma \frac{d}{ds}(\rho r'')\right] + \frac{d}{ds}\left(\frac{\sigma}{\rho} r'\right) + \frac{\rho}{\sigma} r''=0$.

NOTES

4.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

NOTES

BLOCK II: HELICES, HELICOIDS AND FAMILIES OF CURVES

UNIT-V HELICES

Structure

5.1 Introduction

This chapter explains the brief discussion of the properties of a wide class of space curves known as helices.

5.2 Objectives

After going through this unit, you will be able to:

- Define cylinderical helix and the axis of the helix
- Derive the properties of helix
- Define circular helix
- Solve the problems in helices

5.3 Helices

Cylinderical helices:

It is a space curve which lies on a cylinder and cuts a generator at a constant angle α with a fixed line is known as the generator (or) axis.

Spherical helix:

If a curve on a sphere is a helix, then the curve is a spherical helix.

Theorem 2.13 *A characteristic property of a helix is that a ratio of a curvature to the torsion is constant.*

Proof:

Part-I

Assume that: The curve is a helix.

To prove: $\frac{k}{\tau}$ is a constant.

Let $\bar{a}$ denote a unit vector along the axis of the helix and $\bar{t}$ denote the unit vector along the tangent and α denote the angle between them.

Now, $\bar{t}.\bar{a} = cos\alpha$ ....(1)

$$\overline{t'}.\bar{a} = 0$$
$$\Rightarrow k\bar{n}.\bar{a} = 0$$
$$\Rightarrow k(\bar{n}.\bar{a}) = 0$$
$$\Rightarrow \bar{n}.\bar{a} = 0$$

(ie) $\bar{a}$ is perpendicular to $\bar{n}$.

$\Rightarrow$ a lies in the rectifying plane. $\bar{a}$ makes an angle α with $\bar{t}$, then it makes an angle (90-α) with $\bar{b}$.

NOTES

$\bar{a} = \bar{t}cos\alpha + \bar{b}sin\alpha$ .....(2)
Diff (2) w.r to s,
0=$t'cos\alpha + b'sin\alpha$
$kncos\alpha - \tau nsin\alpha$=0
$\Rightarrow (kcos\alpha - \tau sin\alpha)n$=0
$\Rightarrow kcos\alpha - \tau sin\alpha$=0

$$\Rightarrow kcos\alpha = \tau sin\alpha$$

$$\Rightarrow \frac{k}{\tau} = \frac{sin\alpha}{cos\alpha}$$

$\frac{k}{\tau}$ =constant

Part-II

5.3 Helices

Assume that: $\frac{k}{\tau}$ is a constant.
To prove: The curve is a helix.
Now, $\frac{k}{\tau}$ =constant (say p)

$\Rightarrow \frac{k}{\tau}$ =p
$\Rightarrow k = \tau$p

$$\Rightarrow k\bar{n} = \tau p.\bar{n}$$

$$\Rightarrow \overline{t'} = -p.\overline{b'}$$

$\Rightarrow \overline{t'} + p.\overline{b'}$=0
$\Rightarrow \frac{dt}{ds} + p.\frac{db}{ds}$=0
$\Rightarrow \frac{d}{ds}(t + pb)$=0

Integrating,
$t + pb = \bar{a}$ (constant vector)
Taking scalar product of t with (2),

$$\bar{t}.\bar{t} + p(b.\bar{t}) = \bar{a}.\bar{t}$$

$$\Rightarrow \bar{a}.\bar{t} = 1$$

$$\Rightarrow \bar{a}.\bar{t} = a(constant)$$

$\Rightarrow \bar{t}$ makes a constant angle α with the direction 'a'. [where $\alpha = 0, 2\Pi, 4\Pi, \ldots$.]
The curve is a helix.

Note:

If the curvature and torsion are both constant then the curve is called a circular helix.

Circular helix:

A helix described on the surface of the cylindrical helix is called a circular helix.

Theorem 2.14

A helix of a constant curvature is neccessarily a circular helix.

Proof:

Let $\bar{a}$ be the unit vector along the axis of the helix.
Let $p(\bar{r})$ be any point on the helix and $p_1(\bar{r}_1)$ be the projection of $p(\bar{r})$ on the plane which is perpendicular to 'a' and the projection of c on the plane.
Let s and s_1 be the arc length of the curve c and the projection of c on the plane.
Let α be the angle at which the curve cuts the generator.

$$\Rightarrow s_1 = ssin\alpha$$

Diff w.r to s,

$$s'_1 = sin\alpha$$

NOTES

Also, z=$scos\alpha$

The position vector of any current point p on the helix is $\bar{r} = (x, y, z)$

$$\bar{r} = (x, y, scos\alpha)$$

$$\overline{r'} = (\frac{dx}{ds}, \frac{dy}{ds}, cos\alpha)$$

(ie) $\bar{t} = (\frac{dx}{ds_1}.\frac{ds_1}{ds}, \frac{dy}{ds_1}.\frac{ds_1}{ds}, cos\alpha)$

Diff w. r. to s, then

$$\frac{dt}{ds} = (\frac{d^2x}{dsds_1}sin\alpha, \frac{d^2y}{dsds_1}sin\alpha, 0)$$

$$= [\frac{d}{ds_1}(\frac{dx}{ds})sin\alpha, \frac{d}{ds_1}(\frac{dy}{ds})sin\alpha, 0]$$

$$= [\frac{d}{ds_1}.\frac{dx}{ds_1}.\frac{ds_1}{ds}sin\alpha, \frac{d}{ds_1}.\frac{dy}{ds_1}.\frac{ds_1}{ds}sin\alpha, 0]$$

$t' = (\frac{d^2x}{ds_1^2}sin^2\alpha, \frac{d^2y}{ds_1^2}sin^2\alpha, 0)$

We know that, $k^2 = |t'|^2$

$$= (\frac{d^2x}{ds_1^2})^2 sin^4\alpha + (\frac{d^2y}{ds_1^2})^2 sin^4\alpha$$

$k^2 = [(\frac{d^2x}{ds_1^2})^2 + (\frac{d^2y}{ds_1^2})^2]sin^4\alpha \ldots\ldots(*)$

Since $\alpha = 90°, sin90° = 1$

$k^2 = (\frac{d^2x}{ds_1^2})^2 + (\frac{d^2y}{ds_1^2})^2$ ....(1)

We know that, if k_1 is the curvature of projection curve, then

$k_1^2 = (\frac{d^2x}{ds_1^2})^2 + (\frac{d^2y}{ds_1^2})^2$ ....(2)

Sub (2) in (1)

$$k^2 = k_1^2 sin^4\alpha$$

$$\Rightarrow k = k_1 sin^2\alpha$$

$$\Rightarrow k_1 = \frac{k}{sin^2\alpha} = kcosec^2\alpha$$

⇒ k_1 is a constant.

Since k is constant.

⇒ Radius is constant.

⇒ Helix is the circular cylinder.

⇒ Helix is a circular helix.

5.4 Check your progress

1. Define cylinderical helix
2. Define circular helix
3. Define axis of helix
4. Define spherical curvature

5.5 Summary

The axis is the space curve which lies on a cylinder and cuts a generator at a constant angle α $with\ fixed\ line$

A characteristic property of a helix is that a ratio of a curvature to the torsion is constant.

A helix described on the surface of the cylindrical helix is called a circular helix.

A helix of a constant curvature is neccessarily a circular helix.

5.6 Keywords

Axis: It is a space curve which lies on a cylinder and cuts a generator at a constant angle α with a fixed line is known as the generator (or) axis.
Spherical helix:
If a curve on a sphere is a helix, then the curve is a spherical helix.
Circular helix: A helix described on the surface of the cylindrical helix is called a circular helix.

5.7 Self Assessment Questions and Exercises

1. Prove that the curve r=(au, bu^2, cu^3) is a helix if and only if 3ac=± $2b^2$.
2. Show that if the curve r=r(s) is a helix, then find the curvature and torsion of the curve r_1=ρt+$\int nds$ where ρ, t, n and s refer to the curve r=r(s).
3. If the involutes of a twisted curve are pane curves, then show that the curve is a helix.
4. Show that a necessary and sufficient condition that a curve be a helix is that
$[r^{iv}, r''', r'']$=- $\kappa^5 \frac{d}{ds}(\frac{\tau}{\kappa})$=0.
5. Show that the locus C of the centre of curvature of a circular helix of curvature κ is a coaxial helix. Show that the locus of a centre of curvature of C is the original helix, and prove that the product of the torsion at corresponding points of the two helices is equal to κ^2.
6. Prove that all osculating planes to a circular helix which pass through a given point not lying on the helix have their points of contact in a plane. Show that the same property holds for any curve for which xdy-ydx=cdz, where c is a constant.
7. Show that the helices on a cone of revolution project on a plane perpendicular to the axis of a cone as logarithmic spirals.
8. Find the coordinate of the cylindrical helix whose intrinsic equation are κ=τ=$\frac{1}{s}$.
9. Show that the helix whose intrinsic equation are ρ=$\tau^{-1} = (s^2+4)/\sqrt{2}$ lies upon a cylinder whose cross-section is a catenary.
10. Show that the locus of the center of curvature of a curve is an evolute only when the curve is plane.
11. Find the involutes of a helix.
12. Find the involutes and evolutes of the twisted cubic given by x=u, y=u^2, z=u^3.

5.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

NOTES

UNIT- VI CURVES ON SURFACES

Structure

6.1 Introduction

This chapter explains the concept of curves on surfaces. In this chapter we consider the entire surface as a collection of parts,each part being given a particular parameterisationanthe adjacent parts are relate by a proper parametric transformation. Using these ideas,we shall define the representation of a surface and discuss the properties of curves on surfaces.

6.2 Objectives

After going through this unit, you will be able to:

- Define a surface of a curve
- Derive the parametric equation of a surface
- Define the types of singularities
- Derive the equation of a normal equation

6.3 Curves on surfaces

Surface:

A surface is defined as the locus of a point whose cartesian co-ordinates (x,y,z) are functions of two independent parameter uv say. Thus x=f(u,v), y=g(u,v), z=h(u,v) ....(1) are called parametric (or) freedom equation of a surface.

The parameter u,v take real values and vary in some region D. The representation (1) of the surface is an explicit form.

A surface is defined as the locus of a point whose position vector $\bar{r}$ can be expressed interms of two parameters. Thus an equation of the form $\bar{r} = \bar{r}(u, v)$ ....(2) represent a surface.

Definition 2.15

The represent of a surface $\bar{r} = \bar{r}(u, v)$ and x=f(u,v), y=g(u,v), z=h(u,v) are due to Guass and therefore they are named as Guassian form of the surface.

The parameters u and v are called curvilinear co-ordinates (or) surface co-ordinates of the current point on the surface.

Monge's form of the surface:

If the equation F(x,y,z)=0 to the surface can be represented in the form z=f(x,y). Then this representation is called as monge's form of the surface.

Theorem

NOTES

a) The parametric equation of the surface are not unique.

b) The constraints equation of the surface represents more than the parametric equations some limes.

Proof:

a) Consider the two parametric equations x=u, y=v, z=$u^2 - v^2$ ....(1)

and x=u+v, y=u-v, z=uv ....(2)

From eqn (1), we get

$x^2 - y^2 = z$ ....(3) which represents the whole of cartesian hyperboloid.

b) Let x=ucoshv, y=usinhv, z=u^2 ....(4) be the parametric equations of the surface where u,v take a part of surface as z≥0, since u takes only the real values.

But if we eliminate u and v from eqn (4) we get $x^2 + y^2 = z$, which is the constraints equation of the surface and represent the whole of the hyperboloid.

Class of a surface:

Let the parametric equation of a surface be x=f(u,v), y=g(u,v) and z=h(u,v).

The surface is said to be of class r if the function f,g,h are single valued and continuous and also possess the partial derivatives of r th order.

Note:

If the partial differentiation with respect to the parameters u and v are denoted by the use suffixes 1 and 2 respectively. Then $r_1 = \frac{\partial r}{\partial u}$, $r_2 = \frac{\partial r}{\partial v}$, $r_{11} = \frac{\partial^2 r}{\partial u^2}$, $r_{12} = \frac{\partial^2 r}{\partial u \partial v} = r_{21}$, $r_{22} = \frac{\partial^2 r}{\partial v^2}$

Regular (or) ordinary point and singularities on a surface:

Let the position vector $\bar{r}$ of the point p on a surface be given by r=(x,y,z)

(ie) r={$x(u,v), y(u,v), zu, v)$}

Then $r_1 = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u})$, $r_2 = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v})$

The point p is called regular point (or) ordinary point if $r_1 \times r_2$ ≠0 ....(1)

(ie) if rank of matrix is 2

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \text{.........(2)}$$

(ie) atleast one of the second order determinant does not vanish.

But if $r_1 \times r_2$=0 at a point p we call the point p as a singularity of the surface. There are two types:

Essential singularities:

There are inherent singularities. (ie) These singularities are due to the nature [(or) geometrical features] of the surface and these are independent of the choice of parametric representation.

Example:

The vertex of the cone is essential singularity.

Artificial singularities:

These singularities aries from the choice of particular parametric representation.

Example:

NOTES

The pole ((or) origin) in the plane reffered to polar co-ordinates in an artificial singularities.

Let $\bar{r}$ be the position vector of a point in a plane then reffered to polar co-ordinates (r, θ), we have

r=$(ucos\theta, usin\theta, 0)$, then $r_1 = (cos\theta, sin\theta, 0)$

$$r_2 = (-usin\theta, ucos\theta, 0)$$

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ cos\theta & sin\theta & 0 \\ -usin\theta & ucos\theta & 0 \end{vmatrix}$$

=uk

=0 when u=0

(2) is not satisfied when u=0.

Hence the pole in the plane is artificial singularities.

Definition 2.17

A representation R of a surface s of class r in E $_3$ is a set of points in E $_3$ covered parts $\{v_j\}$, each part v_j being given by parametric equation of class r. Each point lying in the overlap of two points v_i, v_j is such that the change of parameters from those of one part to those of other part is proper and class of r.

R-equivalent:

Two representation R, R' are said to be r-equivalent if the composite family of parts (v_j, v'_j) satisfies the condition that at each point p lying in the overlap of any two points, the change of parameters from those of one part to those os another is proper and class r.

Definition 2.18

A surface s of class r in E_3 is an r-equivalent class of representation.

Transformation of parameters:

The set of parameters u,v expressing the co-ordinates of a point on a surface can be transformed to another set of parameter transformation of the form

U=$\phi(u, v)$, V=$\psi(u, v)$

It is only for those transformed which transform regular points into regular points.

We know that

The condition for regular point in parameters u,v is $r_1 \times r_2 \neq 0$

Now,

$$r_1 = \frac{\partial r}{\partial u}$$

$$= \frac{\partial r}{\partial U} . \frac{\partial U}{\partial u} + \frac{\partial r}{\partial V} . \frac{\partial V}{\partial u}$$

$$r_1 = \frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial u} + \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial u}$$

$$r_2 = \frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial v} + \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial v}$$

$$r_1 \times r_2 = (\frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial u} + \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial u}) \times (\frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial v} + \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial v})$$

$$= (\frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial u} \times \frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial v}) + (\frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial u} \times \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial v}) + (\frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial u} \times \frac{\partial r}{\partial U} . \frac{\partial \phi}{\partial v})$$

$$+ (\frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial u} \times \frac{\partial r}{\partial V} . \frac{\partial \psi}{\partial v})$$

$$= (\frac{\partial r}{\partial U} \times \frac{\partial r}{\partial V})(\frac{\partial \phi}{\partial u} . \frac{\partial \phi}{\partial v}) + (\frac{\partial r}{\partial U} \times \frac{\partial r}{\partial V})(\frac{\partial \psi}{\partial u} . \frac{\partial \psi}{\partial v})$$

$$= \frac{\partial r}{\partial U} \times \frac{\partial r}{\partial V} (\frac{\partial \phi}{\partial u}.\frac{\partial \phi}{\partial v} + \frac{\partial \psi}{\partial u}.\frac{\partial \psi}{\partial v})$$

$$r_1 \times r_2 = (\frac{\partial r}{\partial U} \times \frac{\partial r}{\partial V}) \frac{\partial(\phi,\psi)}{\partial(u,v)} \ldots\ldots (2)$$

NOTES

If $r_1 \times r_2 \neq 0$ then the jacobian $(\frac{\partial r}{\partial V}) \frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ ...(3) in some domain D.

The transformation (1) under the condition (3) is called proper transformation.

Curves on a surface

Curvilinear equation of the curves on the surface:

We know that, the curve is the locus of the point whose position vector $\bar{r}$ can be expressed as a function of a single parameter.

Let us consider a surface r=r(u,v) defined on a domain D and if u and v are functions at single parameter 't' then the position vector r becomes function of single parameter t and hence it is locus is a curve lying on a surface r=r(u,v).

Let u=u(t), v=v(t) then r=r(u(t), v(t)) is a curve lying on a surface r=r(u,v)in D. The equation u=u(t) and v=v(t) are called the curvilinear of the curve on the surface.

Parametric curves:

Let r=r(u,v) be the equation of the surface defined on a domain D.

Now by keeping u=constant (or) v=constant, we get the curves of special importance and are called the parametric curves.

Thus if v=c(say) then as u varies then the point r=r(u,c) describe a parametric curves called **u-curve.**

For u-curve, u is a parameter and determine a sense along the curve. The tangent to the curve in the sense of u increasing is along the vector r_1.

Similarly, the tangent to **v-curve** in the sense v increasing is along the vector r_2.

We have 2 system of parametric curves viz. u-curve and v-curve and since we know that $r_1 \times r_2 \neq 0$

The parametric curve of different systems can't touch each other.

If $r_1.r_2$=0 at a point p, then 2 parametric curves through the point p are orthogonal.

If this condition is satisfied at every point.

(ie) For all values of u and v in the domain D, the two system of parametric curves are orthogonal.

Tangent plane:

Let the equation of the curve be u=u(t), v=v(t) then the tangent is parallel to the vector $\dot{r}$, where

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial u}.\frac{\partial u}{\partial t} + \frac{\partial r}{\partial v}.\frac{\partial v}{\partial t}$$

$$= r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}$$

$$dr = r_1 du + r_2 dv$$

But r_1 and r_2 are non-zero and independent vectors.

The tangent to the curve through a point p on the surface lie in the plane. This plane is called the tangent plane at p.

Tangent line to the surface:

Tangent to the any curve drawn on a surface is called a tangent line to the surface.

Theorem 2.19

NOTES

The equation of a tangent plane at p on a surface with position vector r=r(u,v) is either R=$\bar{r} + a\bar{r}_1 + b\bar{r}_2$ (or) (R-r).($r_1 \times r_2$)=0 where a and b are parameter.

Proof:

Let $\bar{r} = r(u, v)$ be the position vector of a point p on the surface.
The tangent plane at p passes through r and contains the vector r_1 and r_2. So if R is the position vector of any point on the tangent plane at p, then $R - \bar{r}, \bar{r}_1$ and $\bar{r}_2$ are coplanar.
Hence we have R=$\bar{r} + a\bar{r}_1 + b\bar{r}_2$ where a and b are arbitrary constant.
Also, $r_1 \times r_2$ is perpendicular to the tangent plane at p.
Hence $r_1 \times r_2$ is perpendicular to R-r lying in the tangent plane.
(R-r).($r_1 \times r_2$)=0 is another form of the equation of the tangent plane at p.

Definition 2.20

The normal to the surface at p is a line through p and perpendicular to the tangent plane at p.

Since $r_1$and r_2 lie in the tangent plane at p and passes through p_1 the normal is perpendicular to both r_1 and r_2 and it is parallel to $r_1 \times r_2$. The normal at p is fixed by the following convention.
If N denotes the unit normal vector at p, then r_1, r_2 and N should form convention, a right handed system using this convention, we get

$$N=\frac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|}=\frac{\bar{r}_1 \times \bar{r}_2}{H} \text{ where } H=|\bar{r}_1 \times \bar{r}_2|$$

Since $\bar{r}_1 \times \bar{r}_2 \neq 0$, we have H=$|\bar{r}_1 \times \bar{r}_2| \neq 0$
$\Rightarrow NH = \bar{r}_1 \times \bar{r}_2$.

Theorem 2.21

The equation of the normal N at a point p on the surface r=r(u,v) is $\bar{R} = \bar{r} + a(\bar{r}_1 \times \bar{r}_2)$.

Proof:

Let R be the position vector of any point on the normal to the surface at p whose position vector is $\bar{r} = \bar{r}(u, v)$.
Since $\bar{r}_1 \times \bar{r}_2$ gives the direction of the normal and $(\bar{R}.\bar{r})$ lies along the normal $\bar{r}_1 \times \bar{r}_2$ and $\bar{R} - \bar{r}$ are parallel.
We have R-r=a($\bar{r}_1 \times \bar{r}_2$) where a is a parameter.
Hence $\bar{R} = \bar{r} + a(\bar{r}_1 \times \bar{r}_2)$ is the equation of the normal at p.

Theorem 2.22

The proper parametric transformation either leaves every normal unchanged or reverses the direction of the normal.

Proof:

Let $\bar{r} = \bar{r}(u, v)$ ....(1) be the given surface and let the parametric transformation be $u' = \phi(u, v)$ and $v' = \psi(u, v)$ ......(2)
We know that,

$$\frac{\partial r}{\partial u} = \frac{\partial r}{\partial u'}.\frac{\partial u'}{\partial u} + \frac{\partial r}{\partial v'}.\frac{\partial v'}{\partial u} \text{ ...(3)}$$

$$\frac{\partial r}{\partial v} = \frac{\partial r}{\partial u'}.\frac{\partial u'}{\partial v} + \frac{\partial r}{\partial v'}.\frac{\partial v'}{\partial v} \text{ ....(4)}$$

Now, $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = (\frac{\partial u'}{\partial u}.\frac{\partial v'}{\partial v} - \frac{\partial u'}{\partial v}.\frac{\partial u'}{\partial u}).(\frac{\partial r}{\partial u'} \times \frac{\partial r}{\partial v'})$

$\Rightarrow HN = \frac{\partial(u', v')}{\partial(u, v)} H'N'$ .....(5), where HN and $H'N'$ have usual meaning in the two systems of parameter u,v and u', v' respectively.
Since the parametric transform is proper, we have,

$J = \frac{\partial(u\prime,v\prime)}{\partial(u,v)} \neq 0$

We know that, H and H' are always positive. Now N and N' are of same sign if $J > 0$ and are of opposite sign $J < 0$.

Since J is continuous function of u,v of u,v in the whole domain D and J does not vanish in D.

J retains the same sign.

Hence N and N' have the same sign.

1. Obtain the surface equation of sphere and find the singularities, parametric curves, tangent plane at a point and the surface normal.

Solution:

i) A sphere is the surface of revolution of a semicircle lying in xoz plane about the z-axis.

The curve meets the revolution at 2 points if p is any point on the circle lying in the xoz-plane then its equation can be written as x=asinu, y=0. z=acosu where u is the angle made by op with the z-axis.

Here u is called co-lattitude of the point p.

After rotation through an angle v about z-axis.

Let PM be the perpendicular on the xoy-plane.

Then XOM is called longitude of the point p and it is denoted as v.

Hence the position vector p on the sphere is

x=OMcosv; y=OM sinv; z=acosu

=OPcos(90-u)cosv;=OP sinusinv; z=acosu

x=asinucosv; y=asinusinv

The surface equation of the sphere is r=(asinucosv,asinusinv,acosu) where u and v are parameters, $0 \leq u \leq \Pi$, $0 \leq v \leq 2\Pi$

ii) **Singularities:**

We know that, r=(asinucosv,asinusinv,acosu)

r_1=(acosucosv,acosusinv,-asinu)

r_2=(-asinusinv,asinucosv,0)

Hence the matrix

M=$\begin{bmatrix} acosucosv & acosusinv & -asinu \\ -asinusinv & asinucosv & 0 \end{bmatrix}$ At u=0 and u=π all the determinant minors of M are zero.

The rank of the matrix is 0.

Here u=0 and u=π are singularity point.

Since these singularities are due to choices of parameter. They are artificial singularities.

By using $r_1 \times r_2$ the result is same.

iii) **Parametric curves:**

First let us find the parametric curves of the system u=constant. When the co-latitude u is constant, acosu is constant.

Let it be A.

Z=a is a plane parallel to the XOY-plane.

If p is the point of intersection of this plane and sphere, where u is the constant then the locus of p is a small circle.

Hence the parametric curves of the system u=constant is a system of parallel small circles which are called parallels.

When the longitude v=constant the plane ZOM is fixed and the point p where v is the constant is the intersection of the sphere and the plane passing through the centrew of sphere.

Hence the locus of p is a great circle.

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Thus the parametric curve of the system v=constant is a system of great circle which are called merdians.
From subdivision (ii) $r_1 . r_2$=0.
Hence the parametric curves are orthogonal.
iv) Tangent plane:

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ acosucosv & acosusinv & -asinu \\ -asinusinv & asinucosv & 0 \end{vmatrix}$$

$$= i(0 + a^2 sin^2 ucosv) - j(0 - a^2 sin^2 usinv) + k(a^2 cosusinucos^2 v + a^2 sinucosusin^2 v)$$

$$= a^2[sin^2 ucosv\vec{i} + sin^2 usinv\vec{j} + sinucosv\vec{k}]$$

$$r_1 \times r_2 = a^2 sinu[sinucosv\vec{i} + sinusinv\vec{j} + cosv\vec{k}]$$

The equation of the tangent plane is (R-r).$(r_1 \times r_2)$=0
X-x, Y-y, Z-z).(sinucosv, sinusinv, cosv)=0
$\Rightarrow (X - x)(sinucosv) + (Y - y)(sinusinv) + (Z - z)(cosv)$=0
Now, H=$|r_1 \times r_2| = a^2 sinu$
N=$\frac{r_1 \times r_2}{H}$=(sinucosv, sinusinv, cosv)
=$\frac{1}{a}\vec{r}$, where $\vec{r}$ is the position vector of a point on the surface.
The surface normal is the outward drawn normal.

2. Obtain the surface equation of a cone with semi-vertical angle α and find the singularities, the parametric curves, tangent plane at a point and a surface normal.

Solution:

Taking the axis of the cone as the z-axis.
Let p(x,y,z) be any point on the cone.
Draw on PN and PM perpendicular to the axis of the cone and XOY plane.
Let NP=u and p be the angle of rotation of the plane.
XOM=v
Hence x=ucosv, y=usinv, z=ucotα
The parametric representation of the point on the surface of the cone is $\bar{r} = (ucosv, usinv, ucot\alpha)$

$$\bar{r}_1 = (cosv, sinv, cot\alpha)$$

$$\bar{r}_2 = (-usinv, ucosv, 0)$$

Singularities:

Here $\bar{r}_1 \times \bar{r}_2 = \begin{vmatrix} i & j & k \\ cosv & sinv & cot\alpha \\ -usinv & ucosv & 0 \end{vmatrix}$

$$= \bar{\imath}(0 - ucosvcot\alpha) - \bar{\jmath}(0 + usinvcot\alpha) + \bar{k}(ucos^2 v + usin^2 v)$$

$$\bar{r}_1 \times \bar{r}_2 = -ucosvcot\alpha\bar{\imath} - usinvcot\alpha\bar{\jmath} + u\bar{k}$$

$\bar{r}_1 \times \bar{r}_2$=0 when u=0
The vertex of the cone is the only singularity. It is an essential singularity.

Parametric curves:

When u=constant the distance of p from z is also constant. p describe a circle.
Hence when u=constant the system of parametric curves is a system of parallel circles with centre on the z-axis.
When v=constant the plane of rotation through the z-axis makes the constant angle with the x-axis so the parametric curves are the intersection of this plane with the cone along a z-axis.

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Hence when v=constant the parametric curves are the generates of the cone through the origin.
When $r_1 . r_2$=0 the parametric curves are orthogonal.

Tangent plane:

The equation of the tangent plane is (R-r).($r_1 \times r_2$)=0
(X-x, Y-y, Z-z).($-ucosvcot\alpha, -usinvcot\alpha, u$)=0
(X-x)($-ucosvcot\alpha$) $-$ $(Y-y)(-usinvcot\alpha)$ + $(Z-z)u$=0, which is the equation of the tangent plane.

Surface normal:

N=$\frac{r_1 \times r_2}{|r_1 \times r_2|}$

$$|r_1 \times r_2| = \sqrt{u^2cos^2vcot^2\alpha + u^2sin^2vcot^2\alpha + u^2}$$

$$= \sqrt{u^2cot^2\alpha(cos^2v + sin^2v) + u^2}$$
$$= \sqrt{u^2cot^2\alpha + u^2}$$
$$= u\sqrt{cot^2\alpha + 1}$$
$$= u\sqrt{cosec^2\alpha}$$
$$|r_1 \times r_2| = ucosec\alpha$$

Surface of revolution:

A surface generated by the rotation of a plane curve about an axis in its plane is called surface of revolution.

Theorem 2.23

The position vector of any point on a surface of revolution generated by the curve (g(u),0,f(u)) in the XOZ plane is $\bar{r}$=(g(u)cosv,g(u)sinv,f(u)) where v is the angle of rotation about the axis.

Proof:

Take the z-axis as the axis of rotation.
Let (g(u),0,f(u)) be the parametric representation of the generation curve in the XOZ-plane.
Let A be any point on the curve. Then its x-coordinate g(u) gives the distance of A from the z-axis.
When the curve revolves about z-axis. A traces out a circle with radius g(u).
When the plane thro' the z-axis has rotated through an angle v. Let p be the position of the point corresponding to A on the curve after rotation.
Draw PM.PN perpendicular to XOY and XOZ planes.
Then AN=PN=g(u) and OM=PN. If (x,y,z) are the coordinate of p. Then we have,

x=OMcosv=PNcosv=g(u)cosv
y=OMsinv=PNsinv=g(u)sinv
z=PM=f(u)

Hence the position vector of a point p on the surface is $\vec{r} = (g(u)cosv, g(u)sinv, f(u))$
Where the domain of (u,v) is $0 \leq v \leq 2\Pi$ with suitable range for u which depends on the surface.

ii) Parametric curves:

Let p be any point on the surface wit u=constant.
⇒ g(u) is also constant.
The locus of p is a circle with radius g(u) for a complete rotation as v axis from 0 to 2π.

NOTES

The parametric curves u=constant are circles parallel to XOY plane which are called parallel.
Let v=constant.
Since v gives the angle of the plane of rotation in this position, the parametric curves are the curves of intersection of this plane of rotation with the surface.
These curves are called Meridians.
iii) Also $r_1 = (g'cosv, g'sinv, f')$

$$r_2 = (-gsinv, gcosv, 0)$$

$r_1 . r_2$=0
Hence the parametric curves are orthogonal.

iv) Normal:

$$r_1 \times r_2 = (-gf'cosv)i - (f'gsinv)j + gg'k$$

and $|r_1 \times r_2| = g^2(f'^2 + g'^2)$
Hence N=$\frac{r_1 \times r_2}{H}$
N=$\frac{(-f'cosv, -f'sinv + g')}{\sqrt{f'^2 + g'^2}}$

Definition 2.24
The anchor ring is a surface generated by rotating a circle of radius about a line in its plane at a distance b>a from its centre.

Note:
This does not meet the axis of rotation where as in the case of sphere, the curve is a semi-circle meeting the axis of rotation at two point.

Theorem 2.25
The position vector of a point on the anchor ring is $\vec{r} = [(b + acosu)cosv, (b + acosu)sinv, asinu]$ *where (b,0,0) is the centre of the circle and z-axis is the axis of rotation.*

Proof:
Take the axis of revolution as the z-axis and the genrating circle in the XOZ plane with c(b,0,0) on the x-axis.
Let CA makes an angle u with x-axis. The coordinate of A is b+a cosu and the z-coordinate of A is asinu.
A has coordinate (b+acosu,0,asinu)
Let p(x,y,z) be the position of the point A after the generating circle has revolved through an angle v. since the points.
A describe the circle about the z-axis, the distance of p from the z-axis is the radius of the circle given by (b+acosu).
Since it has been revolved through an angle v, its x and y coordinates are (b+acosu)cosv,(b+acosu)sinv.
For the points: A and P its coordinates asinu is always constant.
Hence the position vector of the point p on the anchor ring is $\vec{r}$=((b+acosu)cosv,(b+acosu)sinv,asinu) where $0 < u < 2\pi$ and $0 < v < 2\pi$ when u=constant.
CA is fixed and revolves about the z-axis.
Hence it is a circle on the anchor ring and these curves are parallel. where v=constant, the rotating plane is fixed.
Hence the parametric curve for $0 < u < 2\pi$ is the intersection of the cross section of this plane and the anchor ring.
It is generating the circle. Thus the meridians are circle.

$$\vec{r}_1 = (-asinucosv, -asinusinv, acosu)$$

$\vec{r}_2 = (-(b + acosu)sinv, (b + acosu)scosv, 0)$, since $\vec{r}_1.\vec{r}_2$=0
The parametric curves are orthogonal
$\vec{r}_1 \times \vec{r}_2$=-(b+acosu)(-acosucosv, -acosusinv, asinu)
Since $b > a$ the above vector is negative for the range of values of u and v.
The normal is directed inside the anchor ring.
Since $|\vec{r}_1 \times \vec{r}_2|$ is always positive.
Great circle:
When a plane cuts the sphere we get the radius of sphere = radius of circle.

6.4 Check your progress

- Define surface
- Define class of R
- Define monge's form
- Define tangent surface
- Define surface of revolution

6.5 Summary

A surface is defined as the locus of a point whose position vector $\bar{r}$ can be expressed interms of two parameters. Thus an equation of the form $\bar{r} = \bar{r}(u, v)$ ....(2) represent a surface.

Tangent to the any curve drawn on a surface is called a tangent line to the surface.

If the equation F(x,y,z)=0 to the surface can be represented in the form z=f(x,y). Then this representation is called as monge's form of the surface.

6.6 Keywords

Surface: A surface is defined as the locus of a point whose position vector $\bar{r}$ can be expressed interms of two parameters. Thus an equation of the form $\bar{r} = \bar{r}(u, v)$ ....(2) represent a surface.

Class of R: A representation R of a surface s of class r in E_3 is a set of points in E_3 covered parts $\{v_j\}$, each part v_j being given by parametric equation of class r. Each point lying in the overlap of two points v_i, v_j is such that the change of parameters from those of one part to those of other part is proper and class of r.

Tangent line to the surface:
Tangent to the any curve drawn on a surface is called a tangent line to the surface.

Monge's form of the surface:
If the equation F(x,y,z)=0 to the surface can be represented in the form z=f(x,y). Then this representation is called as monge's form of the surface.

Surface of revolution:
A surface generated by the rotation of a plane curve about an axis in its plane is called surface of revolution

6.7 Self Assessment Questions and Exercises

1. The following surface are given in the parametric form

i)Ellipsoid (a sinu cosv, b sinu sinv, c cosu)
ii)Elliptic paraboloid (au cosv, bu sinv, u^2)
iii)Cone (a sinhu sinhv, b sinhu coshv, c sinhu)

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iv)Cylinder (u cosv, u sinv, u)
v)Plane (u+v, u-v, u).
Obtain in each case the representation of the surface in the form f(x,y,z)=0.

2. Discuss the nature of the points on the following surface.
i)r=(u,v,0)
ii)r=(u cosv, u sinv, 0)
iii)r=(u, v, $\sqrt{1-u^2-v^2}$)
3. Show that on the surface r=(a(u+v), b(u-v), uv), the parametric curves are straight lines.
4. Find the parametric curves u=constant and v=constant on the surface of revolution r=(u cosv, u sinv, f(u))
5. Show that the parametric curves are orthogonal on the surface r=(u cosv, u sinv, alog [u+ $(u^2-\alpha^2)^{\frac{1}{2}}$])
6. Establish the formule
i)[N, r_1, r_2]=H
ii) $r_1 \times \mathrm{N} = \frac{1}{H}(\mathrm{F}r_1 - \mathrm{E}r_2)$
iii) $r_2 \times \mathrm{N} = \frac{1}{H}(\mathrm{G}r_1 - \mathrm{F}r_2)$

6.8 Further Readings

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NOTES

UNIT-VII HELICOIDS

Structure

7.1 Introduction

This chapter deals with screw motion of a surface and representation of helicoids in the surface. Also first fundamental form of a surface, direction ratios and direction cosines are derived. Parametric directions and angle between parametric directions are discussed.

7.2 Objectives

After going through this unit, you will be able to:

- Define screw motion
- Derive the representation of generalized helicoid
- Define direction ratios and direction coefficients
- Derive first fundamental form of a surface.
- Solve the problems related to helicoids.

7.3 Curves on surfaces

Screw motion:
There are surfaces which are generated not only by rotation alone but by a rotation followed by a translation such a motion is called screw motion.
Right helicoid:
The surface generated by the screw motion of the x-axis about the z-axis is called a right helicoid.
Representation of a right helicoid:
This is the helicoid generated by a straight line which meets the axis at right angles. If we take the axis as the generating line it rotates about the z-axis and moves upwards.
Let OP be the translated position of x-axis after rotating through an angle v.
Let (x,y,z) be the coordinates of p. Draw PM perpendicular to XOY plane and let OM=u. Then x=ucosv, y=usinv and z=PM.
By our assumption, the distance PM=z translated by the x-axis is proportional to the angle v of rotation.
(ie) Let $\frac{z}{v}$=a (constant)
The position vector of any point on the right helicoid is

$$\vec{r} = (ucosv, usinv, av)$$

Now, $\vec{r}^1 = (cosv, sinv, 0)$

$$\vec{r}^2 = (-usinv, ucosv, a)$$

$\vec{r}^1 . \vec{r}^2$=0

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The parametric curves are orthogonal. when u=constant then the equation of the helicoid becomes
$\vec{r} = (ccosv, csinv, av)$, which are circular helices on the surface.
The parametric curves v=constant are the generators at the constant distance from the XOY plane.
Now $\vec{r}_1 \times \vec{r}_2$=(asinv, -acosv,u) and H=$\sqrt{a^2+u^2}$
The unit normal N=$\frac{\vec{r}_1\times\vec{r}_2}{|\vec{r}_1\times\vec{r}_2|}$
N=$\frac{(asinv,-acosv,u)}{\sqrt{a^2+u^2}}$

Pitch of helicoid:

If v=2π, then $2\pi a$ is the distance translated after one complete rotation. This is called the pitch of helicoid.

Representation of general helicoid:

The general helicoid is with z-axis as the axis generated by the curve of intersection of the surface with any plane containing z-axis.
Since the section planes of the surfaces by such planes are congruent curves. We can take the plane to be XOZ plane and generate the helicoid.
Thus the equation of the generating curves in the XOZ plane can be taken as x=g(u), y=0, z=f(u).
Let the curve in the XOZ plane rotate about the z-axis through an angle v and let it have the translation proportional to the angle v of rotation which we can take it as av.
Since any point on the curve traces a circle with centre on the z-axis and radius g(u) and z-coordinate is translated through av. The position vector of any point $\bar{r}$ on the general helicoid is

$\vec{r}$=(g(u)cosv, g(u)sinv, f(u)+av)

$$\vec{r}_1 = (g'(u)cosv, g'(u)sinv, f'(u))$$
$$\vec{r}_2 = (-g(u)sinv, g(u)cosv, a)$$

Also $\vec{r}_1.\vec{r}_2 = f'(u).a$
Hence when the parametric curves are orthogonal then either $f'(u)$=0 (or) a=0
If $f'(u)$=0 then f(u)=constant.
The surface if a right helicoid.
If a=0 we don't have screw motion and we have only the rotation about z-axis.
Hence the helicoid is a surface of revolution.
When v=constant, the parametric curves are the various position of the generating curve on the plane rotation.
When u=constant (It follows from the equation of the helicoid) the parametric curves are helices on the surface.

Metric

The first fundamental form:

Let $\bar{r} = \bar{r}(u, v)$ be the given surface. Let the parameters u,v be the function of a single parameter 't'.
Then $\bar{r} = r(u(t), v(t))$ .Hence it represent a curve on the surface with t as parameter.
The arc length interms of parameter t is given by
$(\frac{ds}{dt})^2 = \frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt} = (\frac{d\vec{r}}{dt})^2$ ...(1)

But $\frac{d\vec{r}}{dt} = \frac{\partial r}{\partial u}.\frac{du}{dt} + \frac{\partial r}{\partial v}.\frac{dv}{dt}$ $=r_1.\frac{du}{dt} + r_2.\frac{dv}{dt}$ ....(2)
Sub (2) in(1)
$(\frac{ds}{dt})^2 = (r_1.\frac{du}{dt} + r_2.\frac{dv}{dt})^2$
$= r_1.r_1.(\frac{du}{dt})^2 + 2r_1.r_2.\frac{du}{dt}.\frac{dv}{dt} + r_2.r_2.(\frac{dv}{dt})^2$ ....(3)
Let $E = r_1.r_1 = r_1^2; F = r_1.r_2$ and $G = r_2.r_2.= r_2^2$ ....(4)
Sub (4) in (3)

$$ds^2 = Edu^2 + 2Fdu.dv + Gdv^2$$

This is called the first fundamental form (or) metric on the surface.

Note:1

Let P and Q be the neighbouring points on the surface with the position vector r and r+dx corresponding to the parameter u,v and u+du, v+dv.
If ds denote the length of the elementary arc joining (u,v) and (u+du, v+dv) lying on the surface, then $ds^2 = Edu^2 + 2Fdu.dv + Gdv^2$
(ie) $(\frac{ds}{dt})^2 = E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2$

$$\frac{ds}{dt} = \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2}$$

$$\Rightarrow ds = \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2}.dt$$

Integrating, we get

$$S=\int_{t_0}^{t} \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G(\frac{dv}{dt})^2}.dt$$

Note:2

Where v=constant, the metric it reduces to $ds^2 = Edu^2$. Where u=constant, the metric it reduces to $ds^2 = Gdv^2$.

Theorem 1.1 *The first fundamental form of a surface is a positive definite quadratic form in du,dv.*

Proof:

The quadratic form Q=$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ for the real values of x_1, x_2 is called a positive definite if q>0 for every (x_1, x_2) ≠(0,0).
The condition for q to be positive definite are $a_{12} = a_{21}$, $a_{11} > 0$ and $a_{11}a_{22} - a_{12}^2$ >0 ....(1)
Now we shall verify that,
I=$Edu^2 + 2Fdu.dv + Gdv^2$ ....(2) satisfies the above condition for a positive definite quadratic form.
Since we are concered only with ordinary points $r_1 \times r_2$ ≠0 and $H^2 = |r_1 \times r_2|^2 > 0$

$$\begin{aligned} H^2 &= |r_1 \times r_2|^2 \\ &= |r_1 \times r_2|.|r_1 \times r_2| \\ &= r_1^2.r_2^2 - (r_1r_2)^2 \\ &= EG - F^2 \end{aligned}$$

Since $H^2 > 0$ always we get $EG - F^2 > 0$
Also, from the definition we have E=$r_1r_1 = r_1^2 > 0$, G=$r_2.r_2 = r_2^2 > 0$
Hence the eqn (2) satisfies the condition eqn(1) for the positive definite quadratic form.
We have $Edu^2 + 2Fdu.dv + Gdv^2 > 0$ for all values of $(x_1, x_2) \neq (0,0)$

Theorem 1.2 *The metric is invarient under the parametric transformation.*

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Proof:

Let the parametric representation be $u' = \phi(u, v), v' = \varphi(u, v)$ in parametric representation u', v'.

Let the equation of the surface be $r' = r(u', v')$

$$r_1 = \frac{\partial r}{\partial u'}$$

$$r_1{}' = r_1 = \frac{\partial r}{\partial u}.\frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v}.\frac{\partial v}{\partial v'} \text{ ....(1)}$$

$$r_2{}' = r_2 = \frac{\partial r}{\partial v'}$$

$$=\frac{\partial r}{\partial v}.\frac{\partial v}{\partial v'} + \frac{\partial r}{\partial u}.\frac{\partial u}{\partial u'} \text{ ....(2)}$$

$$(1) \Rightarrow r_1{}' = r_1.\frac{\partial u}{\partial u'} + r_2.\frac{\partial v}{\partial v'} \text{ ....(3)}$$

$$(1) \Rightarrow = r_2{}' = r_1.\frac{\partial v}{\partial v'} + r_2.\frac{\partial u}{\partial u'} \text{ ....(4)}$$

Also du=$\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'$ .....(5)

dv=$\frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv'$ .....(6)

If $E', F' and G'$ are the first fundamental coefficient in the new parametric system then We have,

$$E'du'^2 + 2F'du'.dv' + G'dv'^2$$

$$= r_1{}'^2 du'^2 + 2r_1{}'r_2{}'du'.dv' + r_2{}'^2 dv'^2$$

=$[r_1{}'du' + r_2{}'dv']^2$ ....(7)

Using the eqn (3) and (4) in (7)

(7)$\Rightarrow$ $[(r_1.\frac{\partial u}{\partial u'} + r_2.\frac{\partial v}{\partial v'})du' + (r_1.\frac{\partial v}{\partial v'} + r_2.\frac{\partial u}{\partial u'})dv']^2$=$[r_1(\frac{\partial u}{\partial u'}du' + \frac{\partial v}{\partial v'}dv') + r_2(\frac{\partial v}{\partial v'})du' + \frac{\partial u}{\partial u'})dv']^2$ ....(8)

Sub (5) and (6) in eqn (8)

(8) $\Rightarrow (r_1 du + r_2 dv)^2 = r_1^2 du^2 + 2r_1 r_2 du.dv + r_2^2 dv^2$=$Edu^2 + 2Fdu.dv + Gdv^2$

$$E'du'^2 + 2F'du'.dv' + G'dv'^2 = Edu^2 + 2Fdu.dv + Gdv^2$$

The metric is invariant under parametric transformation.

Theorem 1.3 *If ω is the angle between the parametric curves at the point of intersection then* $tan\omega = \frac{H}{F}$

Proof:

Let P be the point of intersection of the two parametric curves with the position vector 'r'.

Then we have,

$r_1 = \frac{\partial r}{\partial u}$; $r_2 = \frac{\partial r}{\partial v}$ are along the tangent to the parametric curves, u=constant and v=constant

If ω is the angle between the parametric curves then we have $r_1.r_2 = |r_1|.|r_2|cos\omega$ and

$$|r_1 \times r_2| = |r_1| \times |r_2| sin\omega N$$

Since F=$r_1.r_2$ and H=$|r_1 \times r_2|$, E= r_1^2 and G=r_2^2, we have

$cos\omega = \frac{F}{\sqrt{EG}}$ and $sin\omega = \frac{H}{\sqrt{EG}}$

$tan\omega = \frac{sin\omega}{cos\omega} = \frac{H}{\sqrt{EG}} \times \frac{\sqrt{EG}}{F}$

$tan\omega = \frac{H}{F}$

Theorem 1.4 *Prove that ds represent the elementary area PQRS on the surface ds=Hdudv.*

NOTES

Proof:

Let the parametric co-ordinates of P,Q,R and S be (u,v), (u,δu+v),(u+δu, v+δv),(u,v+δv) respectively.

When δu, δv are small, the small position PQRS is a parallelogram.

Now PQ = OQ-OP

$= \vec{r}(u+\delta u, v) - \vec{r}(u,v)$

Using approximate taylor's expansion, we have

PQ = $\vec{r}(u,v) + \frac{\partial r}{\partial u}\delta u - r(u,v)$ $= r_1\delta u$

Similarly, PS=$r_2\delta v$

By replacing $\delta u, \delta v$ by du and dv, the vector area of the parallelogram is $r_1du + r_2dv$

ds =$|\bar{r}_1du \times \bar{r}_2dv|$

=$|\bar{r}_1 \times \bar{r}_2|dudv$

ds =Hdudv

This proves that Hdudv gives the elementary area ds on a surface.

1. Find E,F,G and H for the paraboloid x=u, y=v, z=$u^2 - v^2$

Solution:

Any point on the paraboloid has position vector r=(u,v,$u^2 - v^2$)

Hence r_1=(1,0,2u) and r_2=(0,1,-2v)

E= $r_1.r_1$ =1+4u^2

F=$r_1.r_2$ =-4uv

G=$r_2.r_2$ =$1 + 4v^2$

Also, $\bar{r}_1 \times \bar{r}_2$=(-2u,-2v,1)

Hence H=$|r_1 \times r_2| = \sqrt{4u^2 + 4v^2 + 1}$, which is also equal to $\sqrt{EG - F^2}$

2. Calculate the first fundamental co-efficient and the area of the anchor ring corresponding to the domain 0$\leq u \leq 2\pi$ and 0$\leq v \leq 2\pi$.

Solution:

The position vector of any point on the anchor ring is

$$\vec{r} = \{(b + acosu)cosv, (b + acosu)sinv, asinu\}$$

$$\vec{r}_1 = \{-asinucosv, -asinusinv, acosu\}$$

$$\vec{r}_2 = \{-(b + acosu)sinv, (b + acosu)cosv, 0\}$$

Now,

E= $r_1^2 = r_1.r_1$= $a^2sin^2u(cos^2v + sin^2v) + a^2cos^2v$

E = a^2 ....(1)

Also, F=$r_1.r_2$=0 ...(2)

G=$r_2^2 = r_2.r_2$ =$(b + acosu)^2sin^2v, (b + acosu)^2cos^2v$

G=$(b + acosu)^2$ ....(3)

(1), (2), (3) give the first fundamental co-efficient.

To find: Area

Let us find H

We know that, $H^2 = EG - F^2$ =$a^2(b + acosu)^2$

H = $a(b + acosu)$ ...(4)

Also, we know that,

The elementary area of the surface is Hdudv.

Entire surface area is given by

S =$\int_0^{2\pi}\int_0^{2\pi} Hdudv$

=$\int_0^{2\pi}\int_0^{2\pi} a(b + acosu)dudv$

S =$2\pi a\int_0^{2\pi} (b + acosu)du$

S =$4\pi^2ab$

NOTES

Direction co-efficient on a surface:

Direction on the surface:

The direction of any tangent line to the surface at a point 'p' is called a direction on the surface at a point p.

Note:

The components of a tangential vector at a point p is of the form a=$\lambda r_1 + \mu r_2$ if a=(λ, μ) is the tangential vector at a point p on a surface then its magnitude is

$|a| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{\frac{1}{2}}$ we have a=$\lambda r_1 + \mu r_2$

Hence $|a|^2 = a.a$

=$(\lambda r_1 + \mu r_2).(\lambda r_1 + \mu r_2)$

=$(\lambda^2 r_1^2 + \mu^2 r_2^2 + 2r_1 r_2 \lambda\mu)$

We know that E=r_1^2, F=$r_1 r_2$, G=r_2^2

We get$|a|^2 = E\lambda^2 + G\mu^2 + 2F\lambda\mu$

$\Rightarrow |a| = \sqrt{E\lambda^2 + G\mu^2 + 2F\lambda\mu}$ ....(1)

Direction co-efficient:

Let 'b' be the unit vector along the tangential vector 'a' at a point 'p'. Let the components of 'b' be (l,m).

Therefore b=$lr_1 + mr_2$

The component (l,m) of the unit vector be at the point 'p' along the tangential vector 'a' is called the direction coefficient of 'a'.

Since b=$lr_1 + mr_2$ and $|a|$=1 then by the definition(1), we have,

$El^2 + Gm^2 + 2Flm = 1$ ...(2)

Theorem 1.5 *If (l,m) and (l', m') are the direction coefficient of two directions at a point p on the surface and 'θ' is the angle between the two directions at a point p then we have,*

i) $cos\theta = Ell' + F(lm' + l'm) + Gmm'$

ii) $sin\theta = H(lm' - l'm)$

Proof:

If (l,m) and (l', m') are the direction coefficient of two directions at a same point p on the surface r=r(u,v) then the corresponding unit vectors along these directions at a point p are

a=$lr_1 + mr_2$

$a' = l'r_1 + m'r_2$ ....(1)

Let θ be the angle between these two directions.

Measuring θ from the direction r_1 to r_2 through the smaller angle, we have

$$a.a' = (lr_1 + mr_2).(l'r_1 + m'r_2)$$

= $ll'r_1^2 + 2(lm' + l'm)r_1 r_2 + mm'r_2^2$

= $Ell' + f(lm' + l'm) + Gmm'$ ....(2)

Also, $a.a' = cos\theta$ and $a \times a' = sin\theta N$....(3)

Sub (3) in (2), we have

$$cos\theta = Ell' + f(lm' + l'm) + Gmm'$$

Now $a \times a' = (lr_1 + mr_2) \times (l'^{r_1} + m'^{r_2})$= $lm'(r_1 \times r_2) - l'm(r_1 \times r_2)$

$$a \times a' = (lm' - l'm)(r_1 \times r_2)$$

$$\Rightarrow sin\theta N = a \times a'$$

$$=(lm' - l'm)(r_1 \times r_2)$$

$$=(lm' - l'm)NH$$

$sin\theta == (lm' - l'm)H,$ where H=$\sqrt{EG - F^2}$, when the two directions are orthogonal.

$$cos\theta = 0$$

$$\Rightarrow Ell' + F(lm' + l'm) + Gmm' = o$$

Note:

$$tan\theta = \frac{sin\theta}{cos\theta}$$

$$tan\theta = \frac{(lm' - l'm)H}{Ell' + F(lm' + l'm) + Gmm'}$$

Direction ratio:

If (l,m) are the direction coefficient of a directions at a point p on the surface then the scalars (λ, μ) which are proportional to (l,m) are called direction ratios of that direction.

Result:1

Find the direction coefficient from the direction ratios (λ, μ).

Solution:

Given the direction ratios (λ, μ)

Since (λ, μ) are proportional to (l,m)

Let $\frac{1}{\lambda} = \frac{m}{\mu} = k$(say)

$\Rightarrow l = \lambda k, m = \mu k$ ...(1)

We know that

The direction coefficient satisfy the identity $El^2 + 2Flm + Gm^2$=1 ...(2)

Using (1) in (2), we have

$E(\lambda k)^2 + 2F(\lambda k)(\mu k) + G(\mu k)^2$=1

$E\lambda^2 k^2 + 2F\lambda\mu k^2 + G\mu^2 k^2$=1

k $^2(E\lambda^2 + 2F\lambda\mu + G\mu^2)$=1

$$k^2 = \frac{1}{E\lambda^2 + 2F\lambda\mu + G\mu^2}$$

$$K = \frac{1}{\sqrt{E\lambda^2+2F\lambda\mu+G\mu^2}}$$

$$\Rightarrow (l,m) = \frac{(\lambda, \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$$

Result:2

Prove that the direction cosine of parametric direction v=constant and u=constant are $(\frac{1}{\sqrt{E}}, 0)$ and $(0, \frac{1}{\sqrt{G}})$ respectively.

Solution:

The vector r at the point p is the tangential vector to the parametric curve v=constant passing through the point p.

(ie) $r_1 = 1.r_1 + 0.r_2 \Rightarrow \lambda = 1, \mu = 0$

We know that

(l,m) = $\frac{(\lambda,\mu)}{\sqrt{E\lambda^2+2F\lambda\mu+G\mu^2}}$

(l,m) = $\frac{(1,0)}{\sqrt{E(1)^2+2F(0)+G(0)^2}}$ $=\frac{(1,0)}{\sqrt{E}}$

$=(\frac{1}{\sqrt{E}}, \frac{0}{\sqrt{E}})$

(l,m)=$(\frac{1}{\sqrt{E}}, 0)$

Similarly, u=constant

$$r_2 = 0.r_1 + 1.r_2$$

$$\Rightarrow \lambda = 0, \mu = 1$$

(l,m) =$\frac{(0,1)}{\sqrt{E(0)^2+2F(1).0+G(1)^2}}$ $=\frac{(0,1)}{\sqrt{G}}$

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(l,m)=$(0,\frac{1}{\sqrt{G}})$

Result:3

Find the angle between two tangential directions at a point on the surface interms of direction ratio.

Solution:

Let (λ,μ) and (λ',μ') be the two direction ratios of the directions on the surface.

If (l,m) and (l',m') be the direction cosines of the corresponding direction, then we have,

(l,m) =$\frac{(\lambda,\mu)}{\sqrt{E\lambda^2+2F\lambda\mu+G\mu^2}}$ .....(1)

(l',m') =$\frac{(\lambda\prime,\mu\prime)}{\sqrt{E\lambda\prime^2+2F\lambda\prime\mu\prime+G\mu\prime^2}}$

We know that, $cos\theta = Ell' + F(lm' + l'm) + Gmm'$

and $sin\theta = H(lm' - l'm)$ .....(2)

Sub (1) in (2), we get

$$cos\theta = \frac{E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu'}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}\sqrt{E\lambda'^2 + 2F\lambda'\mu' + G\mu'^2}}$$

$$sin\theta = \frac{H(\lambda\mu' - \lambda'\mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}\sqrt{E\lambda'^2 + 2F\lambda'\mu' + G\mu'^2}}$$

When $\theta = \frac{\pi}{2}, cos\theta = 0$

$$\Rightarrow 0 = E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu'$$

The two different at p cuts orthogonally iff the direction cosines of direction satisfy the condition. $E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0$

Result:4

If l and m are direction cosines of any direction at a point p on the surface. Find the angle between this direction and parametric directions.

Solution:

The direction cosines of parametric direction corresponding to the curve v=constant are $(\frac{1}{\sqrt{E}}, 0)$

We know that, $cos\theta = Ell' + F(lm' + l'm) + Gmm'$

and $sin\theta = H(lm' - l'm)$

Here $l' = \frac{1}{\sqrt{E}}, m' = 0$

If θ_1 is the angle between l,m and the direction v=constant corresponding to $(\frac{1}{\sqrt{E}}, 0)$ then we have,

$$cos\theta_1 = El.\frac{1}{\sqrt{E}} + F(0 + \frac{1}{\sqrt{E}}, m) + Gm.0$$

$$cos\theta_1 = \frac{1}{\sqrt{E}}El + F(\frac{1}{\sqrt{E}}, m) = \frac{1}{\sqrt{E}}(lE + Fm)$$

$$sin\theta_1 = H(0 - \frac{1}{\sqrt{E}}m)$$

$$sin\theta_1 = -H\frac{1}{\sqrt{E}}m = H\frac{1}{\sqrt{E}}|m|$$

Similarly, if θ_2 is the angle between l,m and the direction $(0, \frac{1}{\sqrt{G}})$ corresponding to u=constant.

$$l' = 0, m' = \frac{1}{\sqrt{G}}$$

$$\cos\theta_2 = E.0 + F(l.\frac{1}{\sqrt{G}} + 0) + Gm.\frac{1}{\sqrt{G}}$$

$$=F(\frac{1}{\sqrt{G}}) + Gm.\frac{1}{\sqrt{G}}$$

$$\cos\theta_2 = \frac{1}{\sqrt{G}}(Fl + Gm)$$

$$\sin\theta_2 = H(l.\frac{1}{\sqrt{G}} - 0)$$

$$=\frac{1}{\sqrt{G}}H.l$$

$$\sin\theta_2 = \frac{H|l|}{\sqrt{G}}$$

Theorem 1.6 *If* (l', m') *of the direction coefficient of the line which makes an angle* $\frac{\pi}{2}$ *with the line whose direction coefficient are (l,m) then* $l' = -\frac{1}{H}(Fl + Gm)$ *and* $m' = \frac{1}{H}(El + Fm)$

Proof:

If (l,m) and (l', m') of the direction coefficient of two directions at a point p on the surface then we have,

$\cos\theta = Ell' + F(lm' + l'm) + Gmm'$ ....(1)

$\sin\theta = H(lm' - l'm)$ ....(2)

when $\theta = \frac{\pi}{2}$

Eqn (1) ⇒ $Ell' + F(lm' + l'm) + Gmm'$

$$(El + Fm)l' + (Fl + Gm)m' = 0$$

The above equation is satisfied for $l' = -\alpha(Fl + Gm), m' = \alpha(El + Fm)$ ...(3) for some scalar α.

To find: α

when $\theta = \frac{\pi}{2}$

1=H$(lm' - l'm)$ ...(4)

Sub (3) in (4)

H$[l(\alpha(El + Fm)) + \alpha(Fl + Gm)m]$=1

H$[\alpha El^2 + \alpha lFm + \alpha Flm + \alpha Gm^2]$=1

H$[\alpha El^2 + 2\alpha lFm + \alpha Gm^2]$=1

$H\alpha El^2 + 2\alpha HlFm + \alpha HGm^2$=1

$\alpha(HEl^2 + 2HlFm + HGm^2)$=1

$$\alpha = \frac{1}{H(El^2 + 2lFm + Gm^2)}$$

Since l,m are the direction coefficient of two directions at a point p on the surface then we have,

$El^2 + 2Flm + Gm^2$=1, $\alpha = \frac{1}{H}$

Sub in (3), $l' = \frac{-1}{H}(Fl + Gm)$, $m' = \frac{1}{H}(El + Fm)$

1. Prove that if (l,m) are the direction coefficient of the tangential direction to the curve u=u(t), v=v(t) at a point p on the surface r=r(u,v) then $l = \frac{du}{ds}$ and m=$\frac{dv}{ds}$

Solution:

The unit tangent vector at any point p on the surface is t =$\frac{dr}{ds}$

(ie) t =$\frac{\partial r}{\partial u}.\frac{du}{ds} + \frac{\partial r}{\partial v}.\frac{dv}{ds}$ = $r_1.\frac{du}{ds} + r_2.\frac{dv}{ds}$

NOTES

Since t=$\frac{dr}{ds}$ represents the unit tangent vector at a point p along the tangential direction to the curve, its coordinate its components are ($\frac{du}{ds}, \frac{dv}{ds}$)
Then ($\frac{du}{ds}, \frac{dv}{ds}$) give the direction coefficient of the tangent at a point p on the surface.

(ie) l =$\frac{du}{ds}$ $\quad and \quad m = \frac{dv}{ds}$

2. Find the parametric directions and the angel between the parametric curves.

Solution:

When the parametric curve v=constant the parametric direction has the direction ratio (du,0) by v.

Its direction cosines are (l,m) =$\frac{(du,0)}{\sqrt{Edu^2}} = \frac{(1,0)}{\sqrt{E}}$

Similarly, When the parametric curve u=constant , the direction ratios of the curve are (0,dv)

Its direction cosines are $(l', m') = \frac{(0,dv)}{\sqrt{Gdv^2}} = \frac{(0,1)}{\sqrt{G}}$

Let θ be the angle between the parametric curves. Then

$cos\theta = \frac{F}{\sqrt{EG}}$ and sin$\theta = \frac{H}{\sqrt{EG}}$

When $\theta = \frac{\pi}{2}, cos\theta = 0$

So the condition of orthogonality of parametric curves is F=0.

3. For the cone with vertex at the origin and semi-vertical angle $\boldsymbol{\alpha}$, show that the tangent plane is the same at all point on the generating line.

Solution:

The position vector of any point on the cone with semi-vertical angle α and the axis of the cone as z-axis is

$$\vec{r} = (ucosv, usinv, ucot\alpha)$$
$$\vec{r}_1 = (cosv, sinv, cot\alpha)$$
$$\vec{r}_2 = (-usinv, ucosv, 0)$$

E=$r_1.r_1 = 1 + cot^2\alpha$, F=$r_1.r_2$=0, G=$r_2.r_2 = u^2$

$H^2 = EG - F^2 = u^2cosec^2\alpha$, H=$ucosec\alpha$

Now, $\vec{r}_1 \times \vec{r}_2 = (-ucosvcot\alpha, -usinvcot\alpha, u)$

Hence N=$\frac{\vec{r}_1 \times \vec{r}_2}{H} = (-cosvcos\alpha, -sinvcos\alpha, sin\alpha)$

The surface normal N is independent.

4. For a right helicoid given by (ucosv,usinv,av) determine $(\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{N})$ at a point on the surface and the direction of the parametric curves. Find the direction making an angle $\frac{\pi}{2}$ at a point on the surface with the parametric curve v-constant.

Solution:

Any point on the right helicoid is

$$\vec{r} = (ucosv, usinv, av)$$
$$\vec{r}_1 = (cosv, sinv, 0)$$
$$\vec{r}_2 = (-usinv, ucosv, a)$$

E=$r_1.r_1 = 1$, F=$r_1.r_2$=0, G=$r_2.r_2 = u^2 + a^2$

$$H^2 = EG - F^2 = u^2 + a^2$$

H= $\sqrt{u^2 + a^2}$

Now, $\vec{r}_1 \times \vec{r}_2 = (asinv, -acosv, u)$

Hence N=$\frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{r_1 \times r_2}{H} = \left(\frac{asinv}{\sqrt{u^2+a^2}}, \frac{-acosv}{\sqrt{u^2+a^2}}, \frac{u}{\sqrt{u^2+a^2}}\right)$

Let the component of N be (N_1, N_2, N_3). Then the direction cosines of the parametric curves are $(\frac{1}{\sqrt{E}}, 0)$=(1,0) and $(0, \frac{1}{\sqrt{G}}) = (0, \frac{1}{\sqrt{u^2+a^2}})$

If γ is an angle made by N with the z-axis, then $cos\gamma = \frac{u}{\sqrt{u^2+a^2}}$

If (l', m') is the direction coefficient orthogonal to parametric direction v=constant, then we have $l' = \frac{-1}{H}(Fl + Gm)$, $m' = \frac{1}{H}(El + Fm)$

Sub for l,m,E,F,G and H in the above step we have, $l' = 0, m' = \frac{1}{\sqrt{u^2+a^2}}$, which is the direction of parametric system v=constant.

7.4 Check your progress

- Define screw motion
- Define metric
- Define direction ratio
- Define direction coefficient
- Define parametric direction

7.5 Summary

- Surfaces which are generated not only by rotation alone but by a rotation followed by a translation such a motion is called screw motion.

- The surface generated by the screw motion of the x-axis about the z-axis is called a right helicoid.
- If v=2π, then $2\pi a$ is the distance translated after one complete rotation. This is called the pitch of helicoid.

- If (l,m) are the direction coefficient of a directions at a point p on the surface then the scalars (λ, μ) which are proportional to (l,m) are called direction ratios of that direction.

- The direction coefficient from the direction ratios (λ, μ) was derived.

- The angle between two tangential directions at a point on the surface interms of direction ratio is derived.

- If l and m are direction cosines of any direction at a point p on the surface, then the angle between this direction and parametric directions is established.

7.6 Keywords

Screw motion:
There are surfaces which are generated not only by rotation alone but by a rotation followed by a translation such a motion is called screw motion.
Right helicoid:

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The surface generated by the screw motion of the x-axis about the z-axis is called a right helicoid.

Pitch of helicoid:

If v=2π, then $2\pi a$ is the distance translated after one complete rotation. This is called the pitch of helicoid.

7.7 Self Assessment Questions and Exercises

1.Find r_1, r_2 and also the tangent plane at an arbitrary point on the surface r=(a cosu, a sinu, v).
2. Obtain the first fundamental form on the surface of revolution r=(u cosv, u sinv,f(u)).
3. For the surface r=(a sinu cosv, a sinu sinv, a(cosu+log (tan $\frac{u}{2}$)).
Compute the first fundamental coefficient and unit normal at any arbitrary point on the surface.
4. Taking x,y as parameter, calculate the first fundamental coefficients and the unit normal to the surface, z=$\frac{1}{2}$(a$x^2 + 2hxy + by^2$).
5. Show that the angle between the curve u-v=constant and u=constant on the right helicoid r=(u cosv, u sinv, cv) is $\frac{\pi}{4}$.

7.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010).

UNIT- VIII FAMILIES OF CURVES

NOTES

Structure

8.1 Introduction

So far, we were concerned with a single curve lying on a surface and associated tangential direction.This chapter introduces the families of curves on a surface and explain some basic properties of such families.

8.2 Objectives

After going through this unit, you will be able to:

- Define family of curves.
- Define orthogonal trajectories
- Derive the properties of family of curves
- Define isometric correspondence

8.3 Curves on surfaces

Definition:

Let $\phi(u,v)$ be a single valued function of u,v possessing continuous partial derivative ϕ_1, ϕ_2 which do not vanish.

Then the implicit equation $\phi(u,v)$=c where c is a real parameter gives a family of curves on the surface $\vec{r} = \vec{r}(u,v)$

Properties:

i) Through every point (u,v) on the surface there passes one and only member of the family.

ii) Let $\phi(u_o, v_o) = c_1$ where (u_o, v_o) is any point on the surface. Then $\phi(u_o, v_o) = c_1$ is a member of the family passing through (u_o, v_o). Hence through every point (u_o, v_o) on the surface, there passes one and only one member of the family.

iii) The direction ratios of the tangent to the curve of the family at (u,v) is $(-\phi_2, \phi_1)$.

Theorem

The curve of the family ϕ(u,v)=constant are the solution of the differential equation $\phi_1$du+$\phi_2$dv=0 ......(1)

and conversely a first order differential equation of the form P(u,v)du+Q(u,v)dv=0 .....(2)

where P and q are differential functions which do not vanish simultaneously define a family of curves.

Proof:

Since $\phi_1 = \frac{\partial \phi}{\partial u}$ and $\phi_2 = \frac{\partial \phi}{\partial v}$, we get from (1),

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0 \text{ giving } d\phi = 0$$

Self-Instructional Material

Hence we conclude that ϕ(u,v)=c. Thus as the constant c varies, the curves of the family are the different solutions of the differential equation.

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Conversely let us consider the equation (2). Unless the equation is exact, it is not in general possible to find a single function ϕ(u,v) such that ϕ_1=P and ϕ_2=Q.
However we can find integrating factor λ(u,v) such that $\phi_1 = P\lambda$ and $\phi_2 = Q\lambda$.
Rewriting the equation (2) in the form $\lambda Pdu + \lambda Qdv$=0, we get $\phi_1 du + \phi_2$dv=0, so that the solution of the equation is ϕ(u,v)=c.
Further from (2), $\frac{du}{dv} = -\frac{Q}{P}$ so that the direction ratios of the tangent to the curves of the family at the point P is (-Q,P).

Theorem

For a variable direction at P, $|\frac{d\phi}{ds}|$ is maximum in a direction orthogonal to the curve ϕ(u,v)=constant through P and the angle between (-ϕ_2, ϕ_1) and the orthogonal direction in which ϕ is increasing is $\frac{\Pi}{2}$.

Proof:

Let P(u,v) be any point on the surface. We shall show that ϕ increases most rapidly at P in a direction orthogonal to the curve of the family passing through P. For this, we prove that $\frac{d\phi}{ds}$ has the greatest value in such a direction.
Let (l,m) be any direction through P on the surface. Let μ be the magnitude of the vector ϕ=(-ϕ_2, ϕ_1). Let θ be the angle between (l,m) and the vector ϕ.
Let us take a=$lr_1 + mr_2$, b=$-\phi_2 r_1 + \phi_1 r_2$
We shall find $a \times b$ expressing sinθ in terms of H and μ where $\mu = |b|$
From the definition $|a| = 1$.
We have $|a \times b| = \mu sin\theta$ ....(1)
and $a \times b = (l\phi_1 + m\phi_2)(r_1 \times r_2)$ so that

$$|a \times b| = H(l\phi_1 + m\phi_2) \quad ....(2)$$

Equating (1) and (2), we obtain

$$\mu sin\theta = H(l\phi_1 + m\phi_2) \quad .....(3)$$

Since (l,m) are the direction coefficient of any direction through P, we have

$$l=\frac{du}{ds},\ m=\frac{dv}{ds} \quad .....(4)$$

Using (4) in (3) and simplifying, we get $\mu sin\theta = H\frac{d\phi}{ds}$
Now μ and H are always positive and do not depend on (l,m).
Hence $\frac{d\phi}{ds}$ has maximum value $\frac{\mu}{H}$ when sinθ has maximum value in which case $\theta = \frac{\Pi}{2}$.
In a similar manner, $\frac{d\phi}{ds}$ has minimum value -$\frac{\mu}{H}$, when $\theta = -\frac{\Pi}{2}$.
Since H>0 and $\mu > 0$, the orthogonal direction for which $\frac{d\phi}{ds}$ >0 is such that $\theta = \frac{\Pi}{2}$.
Hence $|\frac{d\phi}{ds}|$ has maximum in a direction orthogonal to $\phi(u,v)$=constant.

Isometric Correspondence:

Theorem

To each direction of the tangent to a curve C at P in S, there corresponds a direction of the tangent to C' at P' in S' and vice-versa.

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Proof:

Let C be a curve of a class ≥1 passing through P ang lying on S. Let it be parametrically represented bu u=u(t) and v=v(t). If S' is the portion corresponding to S under the relation (1) in the preceding paragraph, then C on S will be mapped onto C' on S' passing through P' with the parametric equations

$$u' = \phi\{u(t), v(t)\},\ v' = \psi\{u(t), v(t)\}$$

The direction ratios of the tangent at P to C are $(\dot{u}, \dot{v})$ where $\dot{u} = \frac{du}{dt}$, $\dot{v} = \frac{dv}{dt}$

Now the direction ratios of the tangents at P' to C' are $(\dot{u'}, \dot{v'})$ where

$$\dot{u'} = \frac{du'}{dt} = \frac{\partial\phi}{\partial u}\dot{u} + \frac{\partial\phi}{\partial v}\dot{v}$$
$$\dot{v'} = \frac{dv'}{dt} = \frac{\partial\psi}{\partial u}\dot{u} + \frac{\partial\psi}{\partial v}\dot{v}$$

Solving the above equation for $\dot{u}$ and $\dot{v}$, we get

$\dot{u} = \frac{1}{J}(\dot{u'}\frac{\partial\psi}{\partial v} - \dot{v'}\frac{\partial\phi}{\partial v})$,

$\dot{v} = \frac{1}{J}(\dot{v'}\frac{\partial\phi}{\partial u} - \dot{u'}\frac{\partial\psi}{\partial u})$, where J≠0

which shows that a given direction to a curve C' at P' corresponds to a definite direction at P to C and vice-versa.

Definition:

Two surfaces S and S' are said to be isometric or applicable if there exists a correspondence $u' = \phi$(u,v), $v' = \psi$(u,v) between their parameters where phi and ψ are single valued and $\frac{\partial(\phi,\psi)}{\partial(u,v)}$ ≠0 such that the metric of S is transformed into metric of S'. The correspondence itself is called an isometry.

Example:

Find the surface of revolution which is isometric with the region of the right helicoid.

Proof:

Let S be r=(g(u) cosv, g(u) sinv, f(u)) the surface isometric with the right helicoid S' given by $r' = (u'cosv', u'sinv', av')$

Using the fact that the isometry preserves the metrics, we determine g(u) and f(u) and then indicate the region of correspondence.

$$r_1 = \frac{\partial r}{\partial u} = (g_1(u)cosv, g_1(u)sinv, f_1(u))$$
$$r_2 = \frac{\partial r}{\partial v} = (-g(u)sinv, g(u)cosv, 0)$$

Now E=$r_1.r_1 = g_1^2(u) + f_1^2(u)$, F=$r_1.r_2$=0, G=$r_2.r_2 = g^2$

Hence the metric on S is $(g_1^2(u) + f_1^2(u))du^2 + g^2dv^2$ ....(1)

For the surface S', we have

$$r_1{}' = (cosv', sinv', 0)$$
$$r_2{}' = (-u'sinv', u'cosv', a)$$

Hence $E' = r_1{}'.r_1{}'$=1, $F' = r_1{}'.r_2{}'$=0,

$$G' = r'_2.r'_2 = (u'^2 + a^2)$$

Hence the metric on S' is $du'^2 + (u^2 + a^2)dv'^2$ ....(2)

The problem is to find the transformation from S to S' such that (1) and (2) are identical. Without loss of generality let us take $u' = \phi(u)$, v$' = v$.

Then we have $du' = \phi_1(u)du$, $dv' = dv$ ......(3)

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Using (3) in (2), we get $\phi_1^2 du^2 + (\phi^2 + a^2)dv^2$ .....(2)
(4) is the metric after transformation. Hence (1) and (4) are identical so that we have

$$g^2 = (\phi^2 + a^2),\ g_1^2 + f_1^2 = \phi_1^2 \text{ ....(5)}$$

From the equation (5), we have to obtain f and g eliminating ϕ.
However, we can guess the solution of (5) as follows.
Let us take ϕ(u)=a sinh u and g(u)=a cosh u .....(6)
(6) satisfies $g^2 = \phi^2 + a^2$.
Using (6) in the second equation of (5), we get

$$a^2 sinh^2 u + f_1^2(u) = a^2 cosh^2 u \text{ so that } f_1^2(u) = a^2.$$

Thus $f_1(u) = a$.
Integrating and choosing the constant of integration to be zero, we get f(u)=au.
Hence the surface of revolution is generated by x=a cosh u, y=0, z=au where the generating curve lies in the XOZ plane and the curve in the XOZ plane is a catenary with parameter a and the direction as Z-axis. Such a surface of revolution is known as catenoid.

Intrinsic properties:

Example:
A helicoid is generated by the screw motion of a straight line skew to the axis. Find the curve coplanar with the axis which generates the same helicoid.
Solution:
The helicoid is generated by a straight line which does not meet the axis of rotation, since the z-axis of rotation and the straight line generating the helicoid are skew lines.
Let a be the shortest distance between them and α be the angle between z-axis and the skew line.
We shall find the coordinates of any point P on the generating skew line. Let OP′ be parallel to CP in the YOZ plane where C is the point (a,0,0) at which the skew line meets the x-axis. Hence the coordinates of P' are (u sin$\alpha, ucos\alpha$) where u=OP'. Since CP and OP' are parallel, the coordinates of P are (a, u sin$\alpha, ucos\alpha$) ......(1)
Let us rotate the axis about the z-axis through angle v and translate it through a distance av parallel to the axis. Using the relation between coordinates in (1) and after rotation, we get
X=a cos v- u sinα sinv, Y=a sinv + u sin α cosv
Since the z-coordinates is subjected to only translation, we obtain z=ucosα+cv
Hence the position vector of any point on the helicoid is
(a cosv-u sinα sinv, a sinv+ u sinα cosv, u cos α +cv)........(2)
The required plane curve is the section of the helicoid with the XOZ plane. Since the equation to the XOZ plane is y=0, we get
a sinv+u sin α cosv=0 which gives u=$-\frac{atanv}{sin\alpha}$ ....(3)
Substituting the value of u in (2), we get the equation of the generating curves as

r=(a cosv+a tanv sinv, 0, cv-a cot α tanv)
=(a secv, 0, cv-a cot α tanv) where vis the parameter of the curve.

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Example:

The metric on the surface is $v^2du^2 + u^2dv^2$. Find the family of curves orthogonal to the curve uv=constant and find the metric referred to new parameters so that these two families are parametric curves.

Solution:

From the given metric, the fundamental coefficients are E=v^2, F=0 and G=u^2 ......(1)

Using the given family of curves uv=constant,

$$v\frac{du}{ds} + u\frac{dv}{ds}=0 \text{ so that } \frac{du}{dv} = -\frac{u}{v}$$

Hence the tangential direction at any point in the curve has the direction ratios (-u,v).

Let (du,dv) be the direction ratios of the orthogonal direction.

Applying the orthogonality condition by taking $\lambda = -u$, μ=v, λ'=du, μ'=dv and substituting for E,F,G from (1), we obtain $-uv^2du + u^2vdv$=0 giving the differential equation of the orthogonal trajectory as vdu-udv=0.

Integrating this equation, we have $\frac{u}{v} = c_2$

Hence uv=c_1 and $\frac{u}{v} = c_2$ are the orthogonal curves on the surface with the given metric. To find the metric on the surface with respect to these two families of curves as parametric curves, let us define the parametric transformations as

$$u' = \phi(u,v) = \frac{u}{v},\ v' = \psi(u,v)=uv \text{ ....(2)}$$

Now J=$\frac{\partial(\phi,\psi)}{\partial(u,v)} = \frac{2u}{v}$ ≠0, the parametric transformation is proper. Since the parametric transformation is proper, it transforms the given family of curves and the orthogonal family into two families of pareametric curves.

Using (2), let us express u and v in terms of u' and v'

$$u^2 = u'^2v^2 = u'^2\frac{v\prime^2}{u^2} \text{ or } u^2 = u'v' \text{ ....(3)}$$

$$v^2 = \frac{v\prime^2}{u^2} = \frac{v\prime^2}{u\prime v\prime} = \frac{v\prime}{u\prime} \text{ or } v^2 = \frac{v\prime}{u\prime} \text{ ....(4)}$$

Further using the parameters u', v', let the position vector r be r'. Using the relations (3) and (4), we have

$$r'_1 = \frac{\partial r}{\partial u\prime}$$

$$= \frac{\partial r}{\partial u}.\frac{\partial u}{\partial u\prime} + \frac{\partial r}{\partial v}.\frac{\partial v}{\partial u\prime}$$

$$= r_1(\frac{v\prime}{\partial u\prime}) - r_2(\frac{v\prime}{2vu\prime^2})$$

$$r'_2 = \frac{\partial r}{\partial v\prime} = \frac{\partial r}{\partial u}.\frac{\partial u}{\partial v\prime} + \frac{\partial r}{\partial v}.\frac{\partial v}{\partial v\prime}$$

$$= r_1(\frac{u\prime}{\partial 2u}) + r_2(\frac{1}{2vu\prime})$$

Let us find the metric with respect to the new parametres. Since F=$r_1.r_2$=0, we have,

$$E' = r_1'.r_1'$$

$$= \frac{v\prime^2}{4u^2}.r_1^2 + \frac{v\prime^2}{4u\prime^4v^2}.r_2^2$$

$$= E.\frac{v\prime^2}{4u^2} + \frac{Gv\prime^2}{4v^2u\prime^4}$$

$$= \frac{v\prime}{u\prime}.\frac{v\prime^2}{4u\prime v\prime} + \frac{u\prime v\prime v\prime^2u\prime}{4u\prime^4v\prime}$$

$$= \frac{v\prime^2}{2u\prime^2}$$

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$$G' = r_2{}'.r_2{}'$$
$$= \frac{u\prime^2}{4u^2}.r_1^2 + \frac{1}{4u\prime^2 v^2}.r_2^2$$
$$= E.\frac{u\prime^2}{4u^2} + \frac{G}{4v^2 u\prime^2}$$
$$= \frac{1}{4}\frac{v\prime}{u\prime}.\frac{u\prime^2}{u\prime v\prime} + \frac{u\prime v\prime u\prime}{4u\prime^2 v\prime}$$
$$= \frac{1}{2}$$

Hence the metric referred to the new parametric coordinates is $ds' = \frac{1}{2}\frac{v\prime^2}{u\prime^2}du'^2 + \frac{1}{2}dv'^2$ for which $u'v' = c_1$ and $\frac{v\prime}{u\prime} = c_2$ are the parametric curves on the surface.

8.4Check your progress

- **Define family of curves**
- **Define isometric correspondence**
- **Derive the surface of revolution which is isometric with the region of thr right helicoid.**

8.5 Summary

- Let $\phi(u, v)$ be a single valued function of u,v possessing continuous partial derivative ϕ_1, ϕ_2 which do not vanish.

- Then the implicit equation $\phi(u, v)$=c where c is a real parameter gives a family of curves on the surface $\vec{r} = \vec{r}(u, v)$

- Through every point (u,v) on the surface there passes one and only member of the family.

- Two surfaces S and S' are said to be isometric or applicable if there exists a correspondence $u' = \phi$(u,v), $v' = \psi$(u,v) between their parameters where phi and ψ are single valued and $\frac{\partial(\phi,\psi)}{\partial(u,v)}$ ≠0 such that the metric of S is transformed into metric of S'. The correspondence itself is called an isometry.

- The surface of revolution which is isometric with the region of the right helicoid.

- A helicoid is generated by the screw motion of a straight line skew to the axis. Find the curve coplanar with the axis which generates the same helicoid.

8.5 Keywords

Definition:

Let $\phi(u, v)$ be a single valued function of u,v possessing continuous partial derivative ϕ_1, ϕ_2 which do not vanish.Then the implicit equation $\phi(u, v)$=c where c is a real parameter gives a family of curves on the surface $\vec{r} = \vec{r}(u, v)$

Definition:

Two surfaces S and S' are said to be isometric or applicable if there exists a correspondence $u' = \phi$(u,v), $v' = \psi$(u,v) between their

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parameters where phi and ψ are single valued and $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ such that the metric of S is transformed into metric of S'. The correspondence itself is called an isometry.

8.6 Self Assessment Questions and Exercises

1. If the parametric curves are orthogonal, show that the differential equation of lines on the surface cutting the curves u=constant at a constant angle β is $\frac{du}{dv} = \tan\beta \sqrt{\frac{G}{E}}$.
2. Find the orthogonal trajectories of the parametric curves u=constant on the surface. R=(u+v, 1-uv, u-v).
3. If the curves $du^2 = (u^2+c^2)dv^2$ form an orthogonal system on the surface r=(u cosv, u sinv, φ(v)), determine φ.
4. If $v^2du^2 + u^2dv^2$ is the metric on a surface, find the family of curves orthogonal to the curve $\frac{u}{v}$ = constant and fid the metric referred to the new parameter so that these two families are parametric curves.

8.7 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

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BLOCK III:GEODESIC PARALLELS AND GEODESIC CURVATURES

UNIT IX GEODESICS

Structure

9.1 Introduction

Defining a geodesic on a surface, we shall obtain canonical geodesic equation and its normal property. Also the Christoffel symbol of first and second kinds are developed. Some properties of geodesics and existance theorem are also derived.

9.2 Objectives

After going through this unit, you will be able to:

- Define geodesic
- Derive the differential equation of geodesic
- Define stationary point
- Derive the normal property of geodesic.
- Understand the existance theorem of geodesics

9.3 Geodesics

Geodesic on a surface

Geodesic:

Let A and B be two given points on a surfaces. Let these points be joined by curves lyping on S. Then any curve possessing stationary length for small variation over S is called Geodesic.

***Note*:**

Let α be an arc and S(α) be the length of the arc α joining A and B on the surface then,

S(α) = $\int_0^1 \dot{S}dt = \int_0^1 E\dot{U}^2 + F\dot{u}\dot{v} + G\dot{V}^2 dt,$ where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$\dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

***Stationary*:**

If α is such that variation s(α) is atmost at order ε^2 for all some variation in α for different $\lambda(t)$ and $\mu(t)$. Then $S(\alpha)$ is said to be stationary and α is geodesic.

Theorem* 1 *Differential equation of a geodesic

(OR) A necessary and sufficient condition for a curve u = u(t) and v = v(t) on a surface r = r(u,v) to be a geodesic is that,

$U\frac{\partial t}{\partial \dot{u}} - V\frac{\partial t}{\partial \dot{v}} = 0 \ldots\ldots (1)$, where $U = \frac{d}{dt}\frac{\partial t}{\partial \dot{u}} - \frac{\partial t}{\partial u} = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{u}}$,
$V = \frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial T}{\partial v} = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}} \ldots\ldots (2)$

***Proof*:**
To prove: (2)
This theorem, we have to prove that following lemma.
***Lemma*: 1**
If g(t) is continuous function for $0 < t < 1$ and if $\int_0^1 v(t)g(t)dt = 0 \ldots\ldots\ldots (3)$ for all admissible function v(t) as defined above, then g(t) = 0.
Proofof lemma
Suppose $\int_0^1 v(t)g(t) = 0$ for all admissible function v(t) and g(t)≠ 0, then there exists a t_0 between 0 and 1 such that $g(t_0) \neq 0$
Take $g(t_0) > 0$ Since g(t) is continuous on (0,1) and $t_0\epsilon(0,1)$
There exists a neighborhood (a,b) such that $g(t_0) > 0$ is (a,b), where $0 \leq a < t < b \leq 1$
Now define a function v(t) as follows,

$$V(t) = \begin{cases}(t-a)^3(b-t)^3 \ \ for \ \ a \leq t \leq b \\ 0 \ \ for \ \ 0 \leq t \leq a \ \ and \ \ b < t \leq 1 \ldots\ldots\ldots\ldots (*)\end{cases}$$

Then V(t) is an admissible function in (0,1) so that (3) can be rewritten as
$\int_0^1 v(t)g(t)dt = \int_b^a v(t)g(t)dt + \int_a^b v(t)g(t)dt + \int_b^1 v(t)g(t)dt =$
$0 + \int_a^b (t-a)^3(b-t)^3.g(t)dt + 0 \ldots\ldots (4)$ [since by (*)]
Since $(t-a)^3(b-t)^3 > 0$ in (a,b) and g(t) > 0 for $a < t < b$ from (4), we get,
$\int_0^1 v(t)g(t) > 0$, which is a ⇒⇐ *tohypothesis*
$\int_0^1 v(t)g(t)$ = 0 for all admissible v(t)
Hence our assumption then there exists t_0 such that $g(t_0) \neq 0$ is false.
Therefore, **g(t) = 0 for all t ∈ (0,1)**
Hence proved for lemma.
***Proof of theorem*:**
To prove: (2)
Let $f(u, v, \dot{u}, \dot{v}) = \sqrt{2T}$, where $2T(u, v, \dot{u}, \dot{v}) = \dot{s}^2 = E\dot{u}^2 + F\dot{u}\dot{v} + G\dot{v}^2$
The arc length $S(\alpha) = \int_0^1 \dot{s}dt = \int_0^1 \sqrt{2T}dt = \int_0^1 f(u, v, \dot{u}, \dot{v})dt$
The arc length is $S(\alpha') = \int_0^1 f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}\dot{v} + \varepsilon\dot{\mu})dt$
Hence the variation in S(α) is

$$S(\alpha') - S(\alpha) = \int_0^1 [f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}\dot{v} + \varepsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})]dt \ldots\ldots (5)$$

Using Taylor's theorem for second variable, weget

$$f(u + \varepsilon\lambda, v + \varepsilon\mu, \dot{u} + \varepsilon\dot{\lambda}, \dot{v} + \varepsilon\dot{\mu}) = f(u, v, \dot{u}, \dot{v}) + \varepsilon\lambda\frac{\partial f}{\partial u} + \varepsilon\mu\frac{\partial f}{\partial v} + \varepsilon\dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \varepsilon\dot{\mu}\frac{\partial f}{\partial \dot{v}} + 0(\varepsilon^2) \ldots\ldots\ldots\ldots (6)$$

Using (6) in (5), we get,
$S(\alpha') - S(\alpha) = \varepsilon\int_0^1 [\lambda\frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} + \dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial t}{\partial \dot{v}}]dt + 0(\varepsilon^2) \ldots\ldots\ldots\ldots (7)$
Using integrating by parts in (7)

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$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}dt = \int_0^1 \frac{\partial f}{\partial \dot{u}}d(\lambda) = [\lambda\frac{\partial f}{\partial \dot{u}}]_0^1 - \int_0^1 \lambda.\frac{d}{dt}(\frac{\partial f}{\partial \dot{u}})dt$$

Since $\lambda = 0$, we have $[\lambda\frac{\partial f}{\partial \dot{u}}]_0^1 = 0$ at t=0 and t=1

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}dt = -\int_0^1 \lambda\frac{d}{dt}(\frac{\partial f}{\partial \dot{u}})dt.\ldots.(8)$$

Similarly , $\int_0^1 \dot{\mu}\frac{\partial f}{\partial \dot{v}}dt = -\int_0^1 \mu\frac{d}{dt}(\frac{\partial f}{\partial \dot{v}})dt.\ldots.(9)$

Sub. (8) and (9) in (7)

$S(\alpha') - S(\alpha) = \varepsilon\int_0^1 \left[\lambda\frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} - \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) - \mu\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)\right]dt + 0(\varepsilon^2) = \varepsilon\int_0^1 [\lambda L + \mu M]dt + 0(\varepsilon^2).\ldots..(10)$, where

L = $\frac{\partial f}{\partial u} - \frac{d}{dt}(\frac{\partial f}{\partial \dot{u}})$, M = $\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)$, for the arc to be geodesic on S.

$S(\alpha)$ should be stationary, it is stationary iff the variation $S(\alpha) - S(\alpha')$ is atmost of order ε^2 for all small variations.

Since $\varepsilon > 0$ and $S(\alpha) - S(\alpha')$ is at ε^2.

Equation (10) $\Rightarrow \int_0^1 (\lambda L + \mu M)dt = 0.\ldots.(11)$, for all admissible function λ, μ class 2 in $0 \le t \le 1$ such that $\lambda = \mu = 0$ at t=0 and t=1

Since E, F, G are of class 1, and $\lambda(t), \mu(t)$ are of class 2, function L,M are continuous function satisfying the condition as that of g(t) lemma.

Apply lemma to (11), choosing λ, μ and g as follows,

i). $v(t) = \lambda, \mu = 0\ and\ \ g(t) = L$

Then $\int_0^1 (\lambda L + \mu M)dt = \int_0^1 \lambda L dt = 0$ which implies L=0 by the lemma.

ii). $\lambda = 0, v(t) = \mu\ and\ \ g(t) = M$

Then $\int_0^1 (\lambda L + \mu M)dt = \int_0^1 \mu M dt = 0$ which implies M=0 by lemma.

Hence L=0,M=0 are the differential equations for u(t), v(t)

Since two equation L=0, M=0 are same for all geodesic on the surface.

Since two equation don't involve this two points A, B explicitly.

Let us rewrite L=0, M=0 interms of T.

Since $f = \sqrt{2T}, L = \frac{\partial f}{\partial u} - \frac{d}{dt}(\frac{\partial f}{\partial \dot{u}})$ becomes

L=
$\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u} - \frac{d}{dt}(\frac{1}{\sqrt{2T}}\frac{\partial T}{\partial \dot{u}}) = \frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u} - [\frac{1}{\sqrt{2T}}\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - ((2T)^{-3/2})\frac{dT}{dt}(\frac{\partial T}{\partial \dot{u}})] = \frac{1}{\sqrt{2T}}[\frac{\partial T}{\partial u} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}})] + [\frac{1}{(2T)^{3/2}}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}]$

Since T $\neq$ 0 canceling $\frac{1}{\sqrt{2T}}$ throughout L=0 becomes,

$\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}.\frac{\partial T}{\partial \dot{u}}\ldots\ldots(12)$

Similarly , we get

M= $\frac{1}{\sqrt{2T}}[\frac{\partial T}{\partial v} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}})] + \frac{1}{(2T)^{3/2}}\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}}\frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - \frac{\partial T}{\partial v} = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}}\ldots\ldots(13)$

Equation (12), (13) give differential equation of geodesic.

They can be usually written as

$$u = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}.\frac{\partial T}{\partial \dot{u}}\ldots\ldots(14)$$

$$v = \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}}\ldots\ldots(15)$$

where t$(u, v, \dot{u}, \dot{v}) = \frac{1}{2}[E\dot{u}^2 + F\dot{u}\dot{v} + G\dot{v}^2]$

This complete the proof of (2)

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Next to prove:

1.As necessary and sufficient condition for α to be a geodesic on surface.

To prove necessary condition:

Let α be a geodesic on the surface then u(t),v(t) satisfy the differential equation (2) from second expression of U and V in equation (14,15),

$\frac{u}{v} = \frac{\partial T}{\partial \dot{u}} / \frac{\partial T}{\partial \dot{v}}$, so that $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$, which proves the necessary condition.

Toprove: The sufficient condition we have to prove the following lemma is true for any curve whether it is geodesic or not.

Lemma:

If u and v are in (1), then $\dot{u}u + \dot{v}v = \frac{dT}{dt} \ldots\ldots (16)$

Since each of u and v have two equal expression for it. we prove thus in two cases.

***Case* 1**:

Consider the first expression for u and v.

Since T is homogeneous function of degree two in $\dot{u}$ and $\dot{v}$

By Euler's Theorem,

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T \ldots\ldots (17)$$

Since T is a function of $u, v, \dot{u}, \dot{v}$, we get

$$\frac{dT}{dt} = \frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v} \ldots\ldots (18)$$

Now,

$\dot{u}u + \dot{v}v = \dot{u}(\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial \dot{u}}) + \dot{v}(\frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - \frac{\partial T}{\partial v}) \ldots\ldots (19)$

Also consider, $\frac{d}{dt}(\dot{u}\frac{\partial T}{\partial \dot{u}}) = \dot{u}\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) + \ddot{u}\frac{\partial T}{\partial \dot{u}}$

$$\Rightarrow \dot{u}\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) = \frac{d}{dt}(\dot{u}\frac{\partial T}{\partial \dot{u}}) - \ddot{u}\frac{\partial T}{\partial \dot{u}} \ldots\ldots (20)$$

Similarly, $\dot{v}\frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) = \frac{d}{dt}(\dot{v}\frac{\partial T}{\partial \dot{v}}) - \ddot{v}\frac{\partial T}{\partial \dot{v}} \ldots\ldots (21)$

Substitute (20, 21) in (19),

$$\dot{u}u + \dot{v}v = \frac{d}{dt}(\dot{u}\frac{\partial T}{\partial \dot{u}}) - \ddot{u}\frac{\partial T}{\partial \dot{u}} - \dot{u}\frac{\partial T}{\partial u} + \frac{d}{dt}(\dot{v}\frac{\partial T}{\partial \dot{v}}) - \ddot{v}\frac{\partial T}{\partial \dot{v}} - \dot{v}\frac{\partial T}{\partial v}$$

$= \frac{d}{dt}[\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}] - [\frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v}]$

$= \frac{d}{dt}(2T) - \frac{dT}{dt} = \frac{dT}{dt}$ [by (17,18)]

Case:2

Consider the second expression for u and v

Now,

$\dot{u}u + \dot{v}v = \dot{u}(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}}) + \dot{v}(\frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{v}}) = \frac{1}{2T}\frac{dT}{dt}[\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}}] = \frac{1}{2T}\frac{dT}{dt}.2T = \frac{dT}{dt}$ [by (17)

We have $\dot{u}u + v\dot{v} = \frac{dT}{dt}$

To prove: The sufficient it is enough to prove that condition (11) implies that U and V satisfy.

The geodesic equation (2)

Let u and v satisfy the condition (11).

Assume that: $\dot{u}, \dot{v}$ are not zero for some value of t for if $\dot{u} = \dot{v} = 0$ simultaneously, then

NOTES

$$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v} \ and \ \frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v}$$

$\Rightarrow \frac{\partial T}{\partial \dot{u}} = 0; \frac{\partial T}{\partial \dot{v}} = 0$ simultaneously.

Therefore, condition is trivially satisfied from the given condition.

$\frac{u}{\frac{\partial T}{\partial \dot{u}}} = \frac{v}{\frac{\partial T}{\partial \dot{v}}} = \theta$ (say)

$U = \theta.\frac{\partial T}{\partial \dot{u}} \ and \ V = \theta.\frac{\partial T}{\partial \dot{v}} \ldots. (22)$

We know that $\dot{u}u + \dot{v}v = \frac{dT}{dt}$ [lemma]

Therefore, $\frac{dT}{dt}(\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}})\theta = 2T\theta$

$\Rightarrow \theta = \frac{1}{2T}.\frac{dT}{dt}$ [by (22 and 17)]

Substitute θ in (22), $U = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{u}} V = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}}$,

which are geodesic equation.

Hence [u(t),v(t)] is a point on the geodesic of a surface which proves the theorem.

Theorem:

i). When v = constant c from all values of u a necessary and sufficient condition that the curve v=c is a geodesic is $GG_1 + FG_2 - 2GF = 0$ (i.e)$EE_2 + FE_1 - 2EF_1 = 0$

ii). When u = constant ∀ values of v on a surface, a necessary and sufficient condition that the curve u= c is a geodesic is $GG_1 + FG_2 - 2GF_2 = 0$

Proof:

On the curve v=c , u can be taken as a parameters.

The equation of the curve are u=t of v=constant.

We know that,

A curve on a surface is a geodesic iff,

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0\ldots..(1)$$

To get the condition for the parametric curve v=constant to be a geodesic, we have to find u,v, $\frac{\partial T}{\partial \dot{u}}, \frac{\partial T}{\partial \dot{v}}$ from the definition, we have

$T = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2]$, where E, F and G are function of u,v.

Now,

$\frac{\partial T}{\partial u} = \frac{1}{2}[E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2], \quad \frac{\partial T}{\partial v} = \frac{1}{2}[E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2]$

$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v}; \quad \frac{\partial T}{\partial V} = F\dot{u} + u\dot{v}\ldots.(2)$

According to choice of parameters $\dot{u} = 1, \dot{v} = 0$

Hence, $\frac{\partial T}{\partial u} = \frac{1}{2}E_1, \quad \frac{\partial T}{\partial v} = \frac{1}{2}E_2, \frac{\partial T}{\partial \dot{u}} = E, \frac{\partial T}{\partial \dot{v}} = F\ldots..(3)$

Using (3) , we get,

$$U = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial \dot{u}} = \frac{dE}{dt} - \frac{1}{2}E_1$$

Using the function of derivatives of E, as a function of u,v , we obtain

$$U = \frac{\partial E}{\partial u}.\frac{du}{dt} + \frac{\partial E}{\partial v}.\frac{dv}{dt} - \frac{1}{2}E_1$$

$$= E_1.\dot{u} + E_2\dot{v} - \frac{1}{2}E_1$$

Since $\dot{u} = 1 and \dot{v} = 0$ we have,

NOTES

$U = \frac{1}{2}E_1$……(4),

V=$\frac{d}{dt}(\frac{\partial T}{\partial v}) - \frac{dT}{\partial v} = \frac{\partial F}{dt} - \frac{1}{2}E_2$

$= \frac{\partial F}{\partial U}.\frac{du}{dt} + \frac{\partial F}{\partial v}.\frac{dv}{dt} - \frac{1}{2}E_2$

$= F_1 - \frac{1}{2}E_2\text{.....}(5)$

Substitute (4) (5) in (1)

$\frac{1}{2}E_1F - (F_1 - \frac{1}{2}E_2)E = 0$ which gives $EE_2 - 2EF_1 + FE_1 = 0$

ii). When u=constant

Let V can be taken as a parameter ie. v=t.

Hence $\dot{u} = 0, \dot{v} = 1$ using these two, (2) becomes,

$$\frac{\partial T}{\partial u} = \frac{1}{2}G_1, \frac{\partial T}{\partial v} = \frac{1}{2}G_2, \qquad \frac{\partial T}{\partial \dot{u}} = F, \frac{\partial T}{\partial \dot{v}} = u\text{......}(6)$$

T= $\frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + \dot{u}\dot{v}]$, where E, F and u are function of E as a function of u,v . we obtain

$$U = \frac{\partial E}{\partial u}.\frac{du}{dt} + \frac{\partial E}{\partial v}.\frac{dv}{dt} - \frac{1}{2}E_1$$

Using (6) we get,

$$U = \frac{d}{dt}.\frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial u} = \frac{dF}{dT} - \frac{1}{2}G_1$$

By the formula for derivatives of F as a function of u, v. we obtain

U= $\frac{\partial F}{\partial u}.\frac{du}{dt} + \frac{\partial F}{\partial v}.\frac{dv}{dt} - \frac{1}{2}G_1$

U= $F_1\dot{u} + F_2\dot{v} - \frac{1}{2}G_1$

Using $\dot{u} = 0, \dot{v} = 1$

$$u = F_2 - \frac{1}{2}u\text{.....}(7)$$

Also, we have $v = \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - \frac{\partial T}{\partial v}$ =$\frac{dG}{dt} - \frac{1}{2}G_2$

$= \frac{\partial G}{\partial u}.\frac{du}{dt} + \frac{\partial G}{\partial v}.\frac{dv}{dt} - \frac{1}{2}G_2$

V = $\frac{1}{2}G_2\text{....}(8)$

Using (7),(8) in eqn (1)

$$(F_2 - \frac{1}{2}G_1)G - \frac{1}{2}G_2F = 0$$

$$\Rightarrow GG_1 + FG_2 - 2GF_2 = 0$$

Converse follows by retracing the steps in both (i) and (ii).

Corollary:

When the parametric curves are orthogonal.

1. v= constant is a geodesic iff E_2 =0
2. u=constant is a geodesic iff G_1=0

Since the parametric curves are orthogonal F=0.

Taking F=0 in above theorem we get the above particular cases.

Theorem:

If $\dot{u} \neq 0$ in the neighborhood of a point on geodesic, then taking u(t)=t, the curve v=v(u) is a geodesic iff v satisfies the second order O.D.E $\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + s = 0$ where P,Q, R,S are functions of u, v determined by E, F, and G.

Proof:

We know that,

NOTES

$$T = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$$

To find: U

$\frac{\partial T}{\partial \dot{u}} = E\dot{u} + F\dot{v} = E + \dot{v}F\ldots(1)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = \frac{dE}{dt} + \frac{dF}{dt}\dot{v} + F\ddot{v}$$

$= \frac{\partial E}{\partial u}.\frac{du}{dt} + \frac{\partial E}{\partial v}.\frac{dv}{dt} + [\frac{\partial F}{\partial u}.\frac{du}{dt} + \frac{\partial F}{\partial v}.\frac{dv}{dt}]\dot{v} + F\ddot{v}$

As $\dot{u} = 1$, $\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) = E_1 + (E_2 + F_1)\dot{v} + F_2\dot{v}^2 + F\ddot{v}$

$$\frac{\partial T}{\partial u} = \frac{1}{2}(E_1 + 2F_1\dot{v} + G_1\dot{v}^2), \dot{u} = 1$$

Hence, $U = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial u})$

Therefore,

$$U = F\ddot{v} + (F_2 - \frac{1}{2}G_1)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1\ldots..(2)$$

To find : V

$$\frac{\partial T}{\partial \dot{v}} = F\dot{u} + G\dot{v} = F + G\dot{v}\ldots..(3)$$

Hence $\frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) = (\frac{\partial F}{\partial u}.\dot{u} + \frac{\partial F}{\partial v}\dot{v}) + (\frac{\partial G}{\partial u}.\dot{u} + \frac{\partial G}{\partial v}\dot{v})\dot{v} + G\ddot{v}$

Since $\dot{u} = 1$, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v}$$

So, $V = \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - \frac{\partial T}{\partial v}$

$=F_1 + (F_2 + G_1)\dot{v} + G_2\dot{v}^2 + G\ddot{v} - \frac{1}{2}(E_2 + 2F_2\dot{v} + G_2\dot{v}^2)$

$= G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2\ldots..(4)$

Changing the sign we use the condition of a geodesic in the form

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0\ldots..(5)$$

Substitute (1),(2),(3) and (4)

$$(E + \dot{v}F)[G\ddot{v} + \frac{1}{2}G_2\dot{v}^2 + G_1\dot{v} + F_1 - \frac{1}{2}E_2] - (F + G\dot{v})[F\ddot{v} + (F_2 - \frac{1}{2}G_1)\dot{v}^2 + E_2\dot{v} + \frac{1}{2}E_1] = 0$$

$$\Rightarrow (EG - F^2)\ddot{v} + \frac{1}{2}[GG_1 + FG_2 - 2GF_2]\dot{v}^3 + \frac{1}{2}(G_2E + 3G_1F - 2FF_2 - 2E_2G)\dot{v}^2 + \frac{1}{2}[2G_1E + 2F_1F - 3E_2F - E_1G]\dot{v} + \frac{1}{2}[2EF_1 - E_2E - E_1F] = 0$$

This can be written as,

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0$$

$H^2[\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + s] = 0$, where

P = $\frac{1}{H^2}.\frac{1}{2}[GG_1 + FG_2 - 2GF_2]$

Q= $\frac{1}{H^2}.\frac{1}{2}(G_2E + 3G_1F - 2FF_2 - 2E_2G)$

R= $\frac{1}{H^2}.\frac{1}{2}[2G_1E + 2F_1F - 3E_2F - E_1G]$

S= $\frac{1}{H^2}.\frac{1}{2}[2EF_1 - E_2E - E_1F]$

Hence the equation of the geodesic is given by $\ddot{v} + P\dot{v}^3 + Q\dot{v}^2 + R\dot{v} + s = 0$.

1. Prove that the curves of the family $\frac{v^3}{u^2}$ = constant are geodesic on surface with the metric $v^2du^2 - 2uvdudv + 2u^2dv^2, v > 0, u > 0$

Solution:

$\frac{{v^3}^2}{u} = C$ is geodesic iff $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$

Choosing a parametric 't' a parametric representation of the curve u,v as

$$u = ct^3; v = ct^2 \ldots\ldots (1)$$
$$\dot{u} = 3ct^2; \dot{v} = 2ct \ldots. (2)$$

for a given metric, $T = \frac{1}{2}[v^2\dot{u}^2 - 2uv\dot{u}\dot{v} + 2u^2\dot{v}^2]$

$$\frac{\partial T}{\partial u} = \frac{1}{2}[-2v\dot{u}\dot{v} + 4u\dot{v}^2]$$
$$= \frac{1}{2}[-2(ct^2)(3ct^2)(2ct) + 4(ct^3)(2ct^2)]$$
$$= \frac{1}{2}[-12c^3t^5 + 4(ct^3)(4c^2t^2)]$$
$$= \frac{1}{2}[-12c^3t^5 + 16c^3t^5]$$
$$= \frac{1}{2}[4c^3t^5]$$
$$\frac{\partial T}{\partial u} = 2c^3t^5$$

$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}[v^2.2\dot{u} - 2uv\dot{v}]$

$= \frac{1}{2}[((ct^2)^2).2(3ct^2) - 2(ct^3)(ct^2)(2ct)]$

$= \frac{1}{2}[(c^2t^4).2(3ct^2) - 2(ct^3)(ct^2)(2ct)]$

$= \frac{1}{2}(6c^3t^6 - 4c^3t^6) = \frac{1}{2}(2c^3t^6)$

Therefore, $\frac{\partial T}{\partial \dot{u}} = c^3t^6$

$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}[2v\dot{u}^2 - 2u\dot{u}\dot{v}]$

$= \frac{1}{2}[(2(ct^2)).((3ct^2)^2) - 2(ct^3)(2ct)(3ct^2)] = \frac{1}{2}[2(ct^2)(9c^2t^4) -$
$2ct32ct3ct2 = 12(18c3t6 - 12c3t6) = 12(6c3t6)$

Therefore, $\frac{\partial T}{\partial \dot{u}} = 3c^3t^6$

$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}[-2uv\dot{u} + 2u^2.2\dot{v}] = \frac{1}{2}[-2(ct^3(ct^2)(3ct^2) + 4((ct^3)^2)(2ct)]$

$= \frac{1}{2}[-2(ct^3(ct^2)(3ct^2) + 4((c^2t^6)(2ct)]$

$= \frac{1}{2}[-6c^3t^7 + 8c^3t^7] = \frac{1}{2}[2c^3t^7]$

$$\frac{\partial T}{\partial \dot{v}} = c^3t^7$$

By theorem :1

$U = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - (\frac{\partial T}{\partial u}) = \frac{d}{dt}(c^3t^6) - (2c^3t^5) = c^3.6t^5) - (2c^3t^5 = 4c^3t^5$

$V = \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - (\frac{\partial T}{\partial v}) = \frac{d}{dt}(c^3t^7) - (2c^3t^6) = c^3.7t^6) - (3c^3t^6 = 4c^3t^6$

$$U.\frac{\partial T}{\partial \dot{v}} - V.\frac{\partial T}{\partial u} = 4c^3t^5(c^3t^7) - 4c^3t^6(c^3t^6) = 4c^6t^{12} - 4c^6t^{12} = 0$$

Hence $\frac{v^3}{u^2} = c$ is geodesic.

NOTES

2. Prove that the parametric curves on a surface are orthogonal, the curve v = constant in geodesic provided E is a function of U only and the curve u = constant is geodesic provided G is a function of v only.

Proof:

By theorem (2) (i)

V= constant then the curve.

V= c is a geodesic if $EF_2 + E_1F - 2EF_1 = 0$

By corollary:

(If parametric curves are orthogonal then F=0.)

Since F=0, consequently $F_1 = 0$

Therefore, E is a function of u.

E=0.

Since E, F and F_1 are zero, then $EF_2 + E_1F - 2EF_1 = 0$

Therefore, v=constant is a geodesic .

By theorem: (2) ,(ii)

u=constant then the curve u=c is a geodesic if $GG_1 + G_2F - 2GF_2 = 0$.

Since parametric curves are orthogonal, then F=0.

$F_2 = 0$

Therefore, G is a function of v.

G=0.

Since F,F_2 and G are zero.

Therefore, $GG_1 + G_2F - 2GF_2 = 0$.

Hence, u=constant is a geodesic.

$\boldsymbol{Theorem}$: $\boldsymbol{Canonical\ Geodesic\ Equation}$

If the arc length 'S' is the parameters of the curve then the geodesic equation are,

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0,$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \ldots (1)$$

These are called canonical geodesic equation.

Proof:

Since as t is a parametric which is geodesic for S is a parametric which is also called geodesic.

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = \frac{1}{2\mathrm{T}}\frac{\partial T}{\partial s} - \frac{\partial T}{\partial u'} \ldots\ldots (2)$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{1}{2\mathrm{T}}\frac{\partial T}{\partial s} - \frac{\partial T}{\partial v'} \ldots (3)$$

where $T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$

Since $u' = \frac{du}{ds} = l; v' = \frac{dv}{ds} = m$

Here we know that l,m are direction coefficient. $El^2 + 2Flm + Gm^2 = l \Rightarrow T = \frac{1}{2}(1), \frac{dT}{ds} = 0 \ldots\ldots (4)$

Apply (4) in (2) and (3) we get, $u = 0, v = 0$

Theorem:

i). If the curves on a surface are not parametric curves, then the sufficient condition for a curve to be a geodesic is either U=0, or V=0.

ii). For a parametric curve u=constant to be a geodesic a sufficient condition is U=0 and V=constant to be a geodesic, the sufficient condition in v=0.

Proof:

For (i):

Let S is used as a parameter.

By Previous lemma,

$Uu' + Vv' = \frac{dT}{ds}$,

$Since\ \frac{dT}{ds} = 0, then\ we\ have\ Uu' + Vv' = 0 \ldots. (1)$

If the curves are not parameters curves, $u' \neq 0 and v' \neq 0$ [by (1)]

Then U and V are not independent.

Hence U is a scalar multiplication of v and v is a scalar multiple of U.

So that either U=0 or V=0 is a sufficient condition for a curve to be geodesic.

For (ii):

For a curve to be a geodesic on a surface.

Then the canonical equation

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0 \ldots. (2)$$

If we take parametric curve u=constant , so that $u' = 0, v' = 0$

By using (i), V=0 and conversely

Hence the condition for u=constant to be geodesic is U=0.

Similarly, v=0 is a sufficient condition for the parametric curve v=constant to be a geodesic.

$\boldsymbol{Theorem}$: $\boldsymbol{Geodesic\ on\ surface\ of\ revolution}$:

Three types of geodesic on a surface of revolution. $r = (g(u)\cos v, g(u)\sin v, f(u)) are$

$i). v = \alpha\phi(u, a) + \beta where \alpha, \beta are constant$

ii). Every meridian v=constant.

iii). A parallel u=constant is geodesic iff its radius is stationary.

Proof:

For (i):

Given $r = (g(u)\cos v, g(u)\sin v, f(u)) \ldots.. (1)$

Then $r_1 = (g_1(u)\cos v, g_1(u)\sin v, f(u))$

$$r_2 = (-g(u)\sin v, g(u)\cos v, 0)$$

So,

$E =$

$r_1 . r_1 = \left(g_1(u)\cos v, g_1(u)\sin v, f_1(u)\right) * \left(g_1(u)\cos v, g_1(u)\sin v, f_1(u)\right)$

$$= \left(g_1^2(u)\cos^2 v, +(u)\sin^2 v + (u)\right) = (g_1^2(u)(\cos^2 v + \sin^2 v) + f_1^2(u)$$

$$= g_1^2(u) + f_1^2(u) \ldots.. 2(a)$$

$$F = r_1 . r_2 = \left(g(u)\cos v, g(u)\sin v, f(u)\right) * (-g(u)\sin v, g(u)\cos v, 0)$$

$$F = 0 \ldots\ldots. 2(b)$$

$$G = r_2 . r_2 = (-g(u)\sin v, g(u)\cos v, 0) * (-g(u)\sin v, g(u)\cos v, 0)$$

$$= g^2(u)(\sin^2 v + \cos^2 v) + 0$$

$$G = g^2(u) \ldots. 2(c)$$

Let us consider $u(t) \neq 0, v(t) \neq 0$

By above theorem,

The canonical geodesic equation are given either by U=0 or V=0.

Without loss of generality,

Let us find v=0,

NOTES

Now $T = \frac{1}{2}\left[Eu'^2 + 2Fu'v' + Gv'^2\right] = \frac{1}{2}[(g_1^2 + f_1^2)u'^2 + g^2v'^2]$
Since g and f are function of u only, we get,

$$\frac{\partial T}{\partial v} = 0 \; and \; \frac{\partial T}{\partial v'} = g^2v'$$

Hence $v = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \left(\frac{\partial T}{\partial v}\right) = \frac{d}{ds}(g^2v')$
Then, the canonical geodesic equation $v = 0$
Then, $\frac{d}{ds}(g^2v') = 0 \ldots..(3)$
Integrating on both sides,
$g^2v' = \alpha \ldots.(4)$, where α be arbitrary constant.
Now if the curves is in the direction v increasing.
v is position and $g^2 > 0$.
So that α can be taken to be a positive constant.
Hence $\frac{d}{ds}(g^2v') = 0$ is the differential equation of the geodesic on the surface of revolution.
Let us take $g(u) \neq \alpha$
Now, equation (4), squaring on both sides

$$g^4dv^2 = \alpha^2ds^2 = \alpha^2[Edu^2 + 2Fdudv + Gdv^2]$$
$$= \alpha^2[(g_1^2 + f_1^2)du^2 + 0 + g^2dv^2]$$
$$= \alpha^2[(g_1^2 + f_1^2)du^2] + \alpha^2g^2dv^2$$
$$g^4dv^2 - \alpha^2g^2dv^2 = \alpha^2[(g_1^2 + f_1^2du^2)]$$
$$g^2(g^2 - \alpha^2)d^2v = \alpha^2[g_1^2 + f_1^2du^2 \ldots (5)$$

Then, we have $dv^2 = \frac{\alpha^2}{g^2}\frac{g_1^2+f_1^2}{g^2-\alpha^2}du^2$

$$dv = \pm\frac{\alpha}{g}\sqrt{\frac{g_1^2 + f_1^2}{g^2 - \alpha^2}}du$$

where we have taken both signs, since the curve can be change direction as u,v moves on the curves.
Equation (4), $\int ing$ on both sides,

$$v = \beta \pm \alpha\int \frac{1}{g}\sqrt{\frac{g_1^2 + f_1^2}{g^2 - \alpha^2}}du$$

Then we can write $v = \alpha\phi(u, \alpha) + \beta$, where α, β are constant.
For(ii):
Let α= 0
Since g^2 ≠0, $v' = 0$ [by (4)]
Hence v=constant, which is the equation of meridian.
Then we prove , every meridian is a geodesic on the surface of revolution.
For(iii):
If $g(u) = \alpha$. Then du=0 [by (5)]
Obviously u=constant.

By above theorem,
The parametric curve u=constant is a geodesic iff U=0.
Hence let us find U.
Since $u' = 0$, we have $T = \frac{1}{2}g^2v'^2$
Hence, $\frac{\partial T}{\partial u'} = 0,\ \frac{\partial T}{\partial u} = gg_1v'^2$

So, $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u}$, $U = -gg_1 v'^2 \ldots\ldots (6)$

Since $u' = 0, ds^2 = g^2 d^2 v$, so that, $(\frac{dv}{ds})^2 = \frac{1}{g^2}$

Using equation (7) in equation (6) we get,

$$U = (-gg_1).\frac{1}{g^2}, U = \frac{-g_1}{g}$$

Hence, U=0 iff $g_1 = 0$

But g is the radius of the parameter u=constant on a surface of revolution.

A parallel is a geodesic on the surface of revolution if its radius is stationary.

Theorem:

Any curve u=u(t), v=v(t) on a surface r=r(u,v) is a geodesic iff the principal normal at every point on the curve is normal to the surface.

Proof:

To prove the theorem, we will establish that for a curve on a surface $r = r(u, v)$ to be a geodesic at a point P on the surface $r''.r_1 = 0$ and $r''.r_2 = 0$ showing that the principal normal $r'' = kn$ of the curve is orthogonal to the tangential direction r_1 and r_2 at P so that the principal normal of the curve coincides with the surface normal.

Since we use canonical geodesic equation in establishing the above result, we derive the canonical geodesic equation with the help of the following identities.

$$\frac{\partial T}{\partial \dot{u}} = \dot{r}.r_1; \;\; \frac{\partial T}{\partial \dot{v}} = \dot{r}.r_2$$
$$u(t) = \ddot{r}.r_1; \;\; v(t) = \ddot{r}.r_2$$

To Prove above identities,

Let us consider

$$\dot{r} = \frac{dr}{dt} = \frac{\partial r}{\partial u}.\frac{du}{dt} + \frac{\partial r}{\partial v}.\frac{dv}{dt}$$
$$= r_1.\dot{u} + r_2\dot{v} \ldots\ldots (2)$$
$$\dot{r}.\dot{r} = (r_1.\dot{u} + r_2\dot{v})^2$$
$$= r_1^2.\dot{u}^2 + 2r_1r_2\dot{u}\dot{v} + r_2^2\dot{v}^2$$
$$= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

We can take $T = \frac{1}{2}\dot{r}^2 \ldots . (3)$

Differentiate (3) partially, we get

$\frac{\partial T}{\partial u} = \dot{r}\frac{\partial \dot{r}}{\partial u}$; $\quad \frac{\partial T}{\partial v} = \dot{r}\frac{\partial \dot{r}}{\partial v}$, $\frac{\partial T}{\partial \dot{u}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{u}}$; $\frac{\partial T}{\partial \dot{v}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{v}}$ ....(4)

Using,(2) and differential partially we have,

$$\frac{\partial \dot{r}}{\partial \dot{u}} = r_1 \; and \; \frac{\partial \dot{r}}{\partial \dot{v}} = r_2$$
$$\frac{\partial \dot{r}}{\partial u} = r_{11}\dot{u} + r_{21}\dot{v}\;, \qquad \frac{\partial \dot{r}}{\partial v} = r_{12}\dot{u} + r_{22}\dot{v} \ldots\ldots (5)$$

where $r_{11} = \frac{\partial^2 r}{\partial u^2}$, $r_{12} = \frac{\partial^2 r}{\partial u \partial v}$, $r_{21} = \frac{\partial^2 r}{\partial v \partial u}$, $r_{22} = \frac{\partial^2 r}{\partial v^2}$

Using (5) in (4), we get

$$\frac{\partial T}{\partial u} = \dot{r}[r_{11}\dot{u} + r_{21}\dot{v}], \qquad \frac{\partial T}{\partial v} = \dot{r}[r_{12}\dot{u} + r_{22}\dot{v}]$$
$$\frac{\partial T}{\partial \dot{u}} = \dot{r}.r_1, \; \frac{\partial T}{\partial \dot{v}} = \dot{r}.r_2 \ldots . (6)$$

Let us find U(t) and V(t) as follows, we have,

$$U(t) = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}}) - \frac{\partial T}{\partial u}$$

NOTES

Using (6),

$$U(t) = \frac{d}{dt}(\dot{r}.r_1) - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v}) = \ddot{r}.r_1 + \frac{dr_1}{dt}.\dot{r} - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v}))$$
$$= \ddot{r}.r_1 + \dot{r}(r_{11}\dot{u} + r_{12}\dot{v}) - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v}).....(7)$$

Using $r_{12} = r_{21}$ we get,

$$U(t) = \ddot{r}.r_1$$
$$Also, V(t) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(\dot{r}.r_2) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v})$$
$$= \ddot{r}.r_2 + \dot{r}(r_{21}\dot{u} + r_{22}\dot{v}) - \dot{r}(r_{12}\dot{u} + r_{22}\dot{v})$$

Hence, we have $V(t) = \ddot{r}.r_1.....(8)$

From (7) and (8) we get the required identities(1).

Instead of taking the parameters t, we can use as well the parameters S.

Therefore, replace t by s , we have,

$$U(s) = r''.r_1 and\ V(s) = r''.r_1$$

Hence the canonical geodesic U(s) and V(s) = 0.

This implies $r''.r_1 = 0\ \ and\ \ r''.r_2 = 0.....(9)$

Therefore, (9) shows that r'' is perpendicular to $r_1 and r_2 lying in the tangent plane at P.$

Hence r'' is the surface normal of the geodesic at P.

But $r'' = \frac{d^2r}{ds^2} = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{dt}{ds}r'' = kn.$

Therefore r'' is along the principle normal at every point of the geodesic is normal to the surface at P.

Since all steps are reversible, the converse is also true.

Hence the proof.

Corollary:

A curve on a surface is a geodesic iff the rectifying plane is tangent to the surface.

Proof:

Since the principle normal to the geodesic at any point P is the normal to the surface the binormal lies in the tangent plane at P.

So the rectifying plane at a point of the geodesic is the tangent plane to the surface.

Conversely, if the rectifying plane at a point of a curve on a surface is the tangent plane to the surface at the point then the principle normal to the curve is normal to the surface.

1. Prove that every helix on a cylinder is a geodesic and conversely.

Solution:

By normal property, we shall Prove that: the surface normal to the cylinder coincide with the principle normal to the helix on the cylinder.

The helix is a geodesic on the cylinder.

Let the generators of the cylinder be parallel to a constant vector 'a'.

Let γ be the helix on the cylinder and let P be any point on γ.

Let t, n be tangent and principal normal at P to γ.

Since te helix cuts the generator at a constant angle we have, **t.a = constant** ....(1)

Diff. (1), we have,

$$\frac{d\bar{t}}{ds}.\bar{a} + t.\frac{d\bar{a}}{ds} = 0.....(2)$$

Since a is constant vector

(2)⇒ $kn.a$= 0 ⇒ $n.a$ = 0

(As t.n = 0) The condition n.a = 0 and t.n = 0

NOTES

$\Rightarrow$ that n is perpendicular to a and t.
Therefore, **n is parallel to a×t.**
But a and t are tangential to surface of cylinder.
Therefore, a×t is parallel to surface normal.
Thus a×t gives the direction of both the principal normal to γ and surface normal.
So γ is f=geodesic.

Conversely,

Let us take a geodesic γ on a cylinder.
At every point P on γ, we have **n.a = N.a**
Since 'a' is parallel to the generator of the cylinder N.a = 0
Therefore, **n.a = 0**
Since k $\neq 0, n.a = 0. \Rightarrow kn.a = 0$
$\Rightarrow \frac{dt}{ds}.a = 0$ ....(3)
As 'a' is aconstant vector $t.\frac{da}{ds} = 0$ ....(4)
From (3) and(4) , $\frac{dt}{ds}.a + t.\frac{da}{ds} = \frac{d}{ds}(t.a) = 0$
Integrating we get,
t.a = constant
Hence the geodesic γ cuts the generator at a constant angle.
Therefore, it is a helix.

Normal Property

Let i,j,k = 1,2, then $\sqrt{ijk}$ is defined as , $\Gamma ijk = \frac{1}{2}\{(r_i.r_j)_k + (r_i.r_k)_j - (r_j.r_k)_i\}$ where $(r_j.r_k)_i$ and other two similar symbols stand for partial differential with respect to u,v according as i=1,2.

Note: V$erify$ $\Gamma 122 = \frac{1}{2}r_1.r_{22}$

Now, $\Gamma 122 = \frac{1}{2}\{(r_1.r_2)_2 + (r_1.r_2)_2 - (r_2.r_2)_1\}$

$$= \frac{1}{2}\{\frac{\partial}{\partial v}(r_1.r_2) + \frac{\partial}{\partial v}(r_1.r_2) - \frac{\partial}{\partial u}(r_2.r_2)\}$$

$$= \frac{\partial}{\partial v}(r_1.r_2) - \frac{1}{2}\frac{\partial}{\partial u}(r_2.r_2)$$

$$= r_1.r_{22} + r_2.r_{12} - r_2.r_{21} = r_1.r_{22}$$

In general, $\boldsymbol{\Gamma ijk = \overline{r_i}.r_{jk}}$

Theorem:

If $\Gamma ijk i,j,k = 1,2$, are the christoffel symbol of the first kind, then the geodesic equation are,

$$Eu'' + Fv'' + \Gamma 111u'^2 + 2\Gamma 112u'v' + \Gamma 122v'^2 = 0 \ldots (1)$$

$$Fu'' + Gv'' + \Gamma 211u'^2 + 2\Gamma 212u'v' + \Gamma 222v'^2 = 0 \ldots\ldots (2)$$

Proof:

We know that,

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + (F_2 - \frac{1}{2}G_1)v'^2 = 0 \ldots. (3)$$

$$Fu'' + Gv'' + (F_1 - \frac{1}{2}G_2)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \ldots. (4)$$

Then the christoffel symbol of first kind is $\Gamma ijk = r_i.r_{jk}$

$\Gamma 111 = r_1.r_{11} = \frac{1}{2}\frac{\partial}{\partial u}(r_1^2) = \frac{1}{2}E_1 \ldots. (5)$

$\Gamma 112 = r_1.r_{12} = \frac{1}{2}\frac{\partial}{\partial u}(r_1^2) = \frac{1}{2}E_2 \ldots.. (6)$

Since $r_{12} = r_{21}$ we have,

$\Gamma112 = \Gamma121 = r_1 . r_{21} = \frac{1}{2} E_2 \ldots . (7)$

NOTES

Now, $\Gamma122 = r_1 . r_{22} = r_1 . \frac{\partial r_2^2}{\partial v}$

To find: $r_1 . \frac{\partial {r_2}^2}{\partial v}$

Let us consider $\frac{\partial}{\partial v}(r_2^2) = 2r_2 . \frac{\partial r_2}{\partial u} = 2r_2 . r_{21}$

From above equation we get,

$$-r_1 . r_{22} = \frac{\partial}{\partial v}(r_1 . r_2) - \frac{1}{2}\frac{\partial}{\partial u}(r_2^2) = F_2 - \frac{1}{2}G_1$$

$$\Gamma122 = r_1 . r_{22} = F_2 - \frac{1}{2}G_1 \ldots . (8)$$

Also, $\Gamma222 = r_2 . r_{22} = \frac{1}{2}\frac{\partial}{\partial v}(r_2^2) = \frac{1}{2}G_2 \ldots . (9)$

$$\Gamma221 = r_2 . r_{21} = \frac{1}{2}\frac{\partial}{\partial u}(r_2^2) = \frac{1}{2}G_1$$

Since $r_{12} = r_{21}$ we have,

$$\Gamma221 = \Gamma212 = r_2 . r_{12} = \frac{1}{2}G_1 \ldots \ldots (10)$$

Now, $\Gamma211 = r_2 . r_{11} = r_2 . \frac{\partial r_1}{\partial u}$

$$\frac{\partial}{\partial u}(r_2 . r_1) = r_2 . r_{11} + r_1 . r_{21}$$

$$Therefore, r_2 . r_{11} = \frac{\partial}{\partial u}(r_2 . r_1) - r_1 . r_{21}$$

$$r_2 . r_{11} = F_1 - \frac{1}{2}E_2$$

From (7), $\Gamma211 = F_1 - \frac{1}{2}E_2 \ldots . (11)$

Substitute (5),(6),and (8) in (3) we get(1)

Substitute (9),(10),and (11) we get(2)

Hence the proof.

Theorem:

a)$u'' = -\frac{1}{2H^2}[lu'^2 + 2mu'v' + nv'^2]$ where

$l = (GE_1 - 2FF_1 + FE_2), m = (GE_2 - FG_1),$

$n = (2GF_2 - GG_1 - FG_2)$

b) $v'' = -\frac{1}{2H^2}\left[lu'^2 + 2\mu u'v' + vv'^2\right]$ $where$

$l = (2EF_1 - EE_2 - FE_1);\ \ \mu = (EG_1 - EE_2);$

$v = (EG_2 - 2FF_2 + FG_1)$

Proof:

Solve for $u'' and\ v''$ from the geodesic equation (I) and (II)

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \ldots . (1)$$

$$Fu'' + Gv'' + (F_1 - \frac{1}{2}E_2)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \ldots . (2)$$

Now, (1). G - (2).F gives,

$$(EG - F^2)u'' + \frac{1}{2}(GE_1 - 2FF_1 + FG_2)u'^2 + (GE_2 - FG_1)u'v' + \frac{1}{2}(2GF_2 - GG_1 - FG_2)v'^2 = 0 \ldots . (3)$$

Replacing the coefficient of $u'^2, v'u'\ \ and\ \ v'^2$ by l, m, n respectively.

We obtain $u'' = -\frac{1}{2H^2}[lu'^2 + 2mu'v' + nv'^2]$

Similarly solve for v''

NOTES

Now, (2).E - (1).F gives,

$$H^2v'' + \frac{1}{2}(2EF_1 - EE_2 - FE_1)u'^2 + (EG_1 - FE_2)u'v' + \frac{1}{2}(EG_2 - 2FF_2 + FG_1)v'^2 = 0$$

Using λ, μ and v for coefficient of $u'^2, u'v', v'^2$

We obtain, $v'' = -\frac{1}{2H^2}[\lambda u'^2 + 2\mu u'v' + vv'^2]$

Differential equation of geodesic using Normal Property:

The normal property of geodesic is given by the identities $r''.r_1 = 0 and\ r''.r_2 = 0$

Using the equation of a surface r=r(u,v).

We shall express the normal property interms of r and its partial derivatives and establish how the new equation derives from the normals property is equivalent to the canonical geodesic equation derived earlier.

We also the christoffel symbols to express the new equations elegantly.

Theorem:

The geodesic equations are $Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0$

$$Fu'' + Gv'' + (F_1 - \frac{1}{2}E_2)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0$$

Proof:

Let the equation of the surface be r=r(u,v), where u=u(s) and V=v(s)

Now, $r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}.\frac{du}{ds} + \frac{\partial r}{\partial v}.\frac{dv}{ds}$

$$r' = r_1.u' + r_2v' \ldots.(1)$$

Differentiate (1) w.r.t S,

$$r'' = \frac{dr'}{ds} = \frac{\partial r'}{\partial u}.\frac{du}{ds} + \frac{\partial r'}{\partial v}.\frac{dv}{ds}$$

$$= (r_{11}u' + r_{12}v')u' + r_1u'' + (r_{21}u' + r_{22}v')v' + r_2v'' \ldots.(2)$$

From the normal property,

$$r''.r_1 = 0 \text{ and } r''.r_2 = 0 \ldots.(3)$$

Taking the scalar product of (2) with $r_1 and r_2$ respectively and using (3) we obtain,

$$r_1.r_1u'' + r_2.r_1.v'' + r_{11}.r_1u'^2 + 2r_{12}.r_1u'v' + r_{22}r_1v'^2 = 0 \ldots.(4)$$

$$r_1.r_2u'' + r_2.r_2.v'' + r_{11}.r_2u'^2 + 2r_{12}.r_2u'v' + r_{22}r_1v'^2 = 0 \ldots.(5)$$

We shall rewrite (4) and(5) using the first fundamental coefficient and their partial derivatives for the coefficients of $u'', v'', u'^2 and v'^2 and u'v'$ as follows,

Now, $r_1.r_1 = E,\ r_1.r_2 = F,\ r_2.r_2 = G$

$$r_1.r_{11} = \frac{1}{2}\frac{\partial}{\partial u}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial u}$$

$$r_1.r_{11} = \frac{1}{2}E_1 \ldots.(6)$$

$$r_1.r_{12} = \frac{1}{2}\frac{\partial}{\partial v}(r_1^2) = \frac{1}{2}\frac{\partial E}{\partial v}$$

$$r_1.r_{12} = \frac{1}{2}E_2 \ldots..(7)$$

$$r_2.r_{22} = \frac{1}{2}\frac{\partial}{\partial v}(r_2^2) = \frac{1}{2}\frac{\partial G}{\partial v}$$

$$r_2.r_{22} = \frac{1}{2}G_2 \ldots.(8)$$

NOTES

$$r_2.r_{21} = \frac{1}{2}\frac{\partial}{\partial u}(r_2^2) = \frac{1}{2}\frac{\partial G}{\partial u}$$

$$r_2.r_{21} = \frac{1}{2}G_1 \ldots.(9)$$

Further $\frac{\partial}{\partial u}(r_2.r_1) = r_2.r_{11} + r_1.r_{21}$ *which gives*

$$r_2.r_{11} = F_1 - \frac{1}{2}E_2 \ldots..(10)$$

$$\frac{\partial}{\partial v}(r_2.r_1) = r_2.r_{12} + r_1.r_{22} \; which \; gives$$

$r_1.r_{22} = F_2 - \frac{1}{2}G_1 \ldots.(11)$

Using (6),(7) and (11) in (4), we have,

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \ldots.(I)$$

$$Uisng(8), (9) \; and \; (10) \; in \; (5) \; we \; have,$$

$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \ldots..(II)$$

Theorem:

The equation (I) and (II) of above theorem are the same as the canonical geodesic equations.

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \left(\frac{\partial T}{\partial u}\right) = 0 \ldots..(1)$$

$$V = \frac{d}{ds}(\frac{\partial T}{\partial v'}) - (\frac{\partial T}{\partial v}) = 0 \ldots.(2)$$

Proof:

Let $T = \frac{1}{2}\left[Eu'^2 + 2Fu'v' + Gv'^2\right]$, *then we have*

$$\frac{\partial T}{\partial u'} = Eu' + Fv' \ldots.(3)$$

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) = \left(\frac{\partial E}{\partial u}.u' + \frac{\partial E}{\partial v}.v'\right)u' + Eu'' + \left(\frac{\partial F}{\partial u}.u' + \frac{\partial F}{\partial v}.v'\right)v' + Fv''$$

$$= E_1u'^2 + E_2u'v' + Eu'' + F_1u'v' + F_2v'^2 + Fv'' \ldots.(4)$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2] \ldots.(5)$$

Further $\frac{\partial T}{\partial v'} = Fu' + Gv' \ldots..(6)$

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) = F_1u'^2 + F_2u'v' + Fu'' + G_1u'v' + G_2v'^2 + Gv'' \ldots..(7)$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}[E_2u'^2 + 2F_2u'v' + G_2v'^2] \ldots..(8)$$

Hence, using equation (4) and(5) we have,

$$\frac{d}{ds}(\frac{\partial T}{\partial u'}) - (\frac{\partial T}{\partial u}$$

$$= Eu'' + Fv'' + E_2u'v' + F_1u'v' + \frac{1}{2}E_1u'^2 - F_1u'v' + (F_2 - \frac{1}{2}G_1)v'^2$$

So that equation (1) becomes

$$Eu'' + Fv'' + E_2u'v' + \frac{1}{2}E_1u'^2 + (F_2 - \frac{1}{2}G_1)v'^2 = 0$$

Similarly, using equation (7) and (8) we get,

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right)-\left(\frac{\partial T}{\partial v}\right)$$
$$= Fu'' + Gv'' + G_1u'v' + \frac{1}{2}G_2v'^2 + (F_1 - \frac{1}{2}G_2)v'^2 = 0$$

Here equation (2) becomes,

$$Fu'' + Gv'' + (F_1 - \frac{1}{2}G_2)v'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0$$

This prove that equation (I) and (II) are equivalent to (1) and (2).

Definition:

The christoffel's symbols of the second kind denoted by Γjk^1 $for\ i, j, k = 1,2$ are defined as

$$\Gamma jk^1 = H^{-2}(G\Gamma 1jk - F\Gamma 2jk)$$
$$\Gamma jk^2 = H^{-2}(E\Gamma 2jk - F\Gamma 1jk)$$

Theorem:

If Γjk^1 and Γjk^2 are christoffel symbol of the second kind, the geodesic equations are $u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 = 0 \ldots.(1)$

$$v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 = 0\ldots.(2)$$

Proof:

We know that, the geodesic equation are,

$$H^2u'' + \frac{1}{2}(GE_1 - 2FF_1 + FE_2)u'^2 + (GE_2 - FG_1)u'v'$$
$$+ \frac{1}{2}(2GF_2 - GG_1 - FG_2)v'^2 = 0 \ldots.(3)$$
$$and\ H^2v'' + \frac{1}{2}(2EF_1 - EE_2 - FE_1)u'^2 + (EG_1 - FE_2)u'v' + \frac{1}{2}(EG_2 - 2FF_2 + FG_1)v'^2 = 0\ldots.(4)$$

Using christoffel symbol of 2^m kind we find the coefficient of different derivatives of the above geodesic equation,

$\Gamma 11^1 = H^{-2}(G\Gamma 111 - F\Gamma 211) = H^{-2}[\frac{1}{2}GE_1 - F(F_1 - \frac{1}{2}E_2)]$

$\Gamma 11^1 = H^{-2}[\frac{1}{2}GE_1 - FF_1 + \frac{1}{2}FF_2]\ldots.(5)$, (since $\Gamma 111 = \frac{1}{2}E_1; \Gamma 211 = F_1 - \frac{1}{2}E_2$)

$$\Gamma 12^1 = H^{-2}(G\Gamma 112 - F\Gamma 112)\Gamma 12^1$$
$$= H^{-2}[\frac{1}{2}GE_2 - \frac{1}{2}FG_1]\ldots..(6)\Gamma 22^1 = H^{-2}(G\Gamma 122 - F\Gamma 222)$$
$$= H^{-2}\left[G\left(F_2 - \frac{1}{2}G_1\right) - \frac{1}{2}FG_2\right]$$
$$\Gamma 22^1 = H^{-2}[GF_2 - \frac{1}{2}GG_1 - \frac{1}{2}FG_2]\ldots.(7)$$
$$\Gamma 11^2 = H^{-2}(E\Gamma 211 - F\Gamma 111)$$
$$= H^{-2}\left[E\left(F_1 - \frac{1}{2}E_2\right) - \frac{1}{2}FE_1\right]$$
$$\Gamma 11^2 = H^{-2}[EF_1 - \frac{1}{2}EE_2 - \frac{1}{2}FE_1]\ldots..(8)$$
$$\Gamma 12^2 = H^{-2}(E\Gamma 212 - F\Gamma 112)\Gamma 12^2$$
$$= H^{-2}[\frac{1}{2}EG_1 - \frac{1}{2}FE_2]\ldots.(9)$$
$$\Gamma 22^2 = H^{-2}(E\Gamma 222 - F\Gamma 122)$$
$$= H^{-2}\left[\frac{1}{2}EG_2 - F\left(F_2 - \frac{1}{2}G_1\right)\right]$$

NOTES

$$\Gamma 11^1 = H^{-2}[\tfrac{1}{2}EG_2 - FF_2 + \tfrac{1}{2}FG_1] \ldots . (10) \quad , \quad \text{Since} \quad H^{-2} \neq 0$$

substitute (5),(6) and(7) in (3) we get the equation (1) also substitute (8) (9) (10) in (4), we get equation (2).

Theorem:

A geodesic can be found to pass through any given point and have any given direction on a surface. the geodesic is uniquely determined by the initial condition.

Proof:

To prove this theorem, we have to derive the second order differential equation and deduce the existence of geodesic at a point from the uniqueness of solution of initial value problem of such as differential equation.

Now, we have $\frac{dv}{du} = \frac{dv}{ds}.\frac{ds}{du} \, and \, \frac{d^2v}{du^2} = \frac{d}{du}\left(\frac{dv}{ds}.\frac{ds}{du}\right)$

$$= \frac{d}{ds}\left(\left(\frac{dv}{ds}.\frac{ds}{du}\right)\frac{ds}{du}\right) = \frac{d^2v}{ds^2}(\frac{ds}{du})^2 + \frac{dv}{ds}.\frac{d}{du}.\left(\frac{ds}{du}\right)$$

$$= \frac{d^2v}{ds^2}(\frac{ds}{du})^2 + \frac{dv}{ds}.\frac{d}{du}.\left(\frac{1}{u}\right) = \frac{d^2v}{ds^2}(\frac{ds}{du})^2 - \frac{1}{u'^2}.u''.\frac{ds}{du}.\frac{dv}{ds}$$

$$\frac{d^2v}{du^2} = \frac{d^2v}{ds^2}(\frac{ds}{du})^2 - \frac{1}{u'^2}.(\frac{ds}{du})^2.\frac{dv}{du} \ldots . (1)$$

We know that,

$$u'' = -\left[lu'^2 + 2mu'v' + nv'^2\right] \quad \ldots . . (2)$$

$$v'' = -[\lambda u'^2 + 2\mu u'v' + \gamma v'^2] \ldots . (3)$$

Now (3) $\times (\frac{ds}{du})^2 -$ (2) $\times \frac{dv}{ds}.(\frac{ds}{du})^2$

$$\Rightarrow v'' * (\frac{ds}{du})^2 - u''(\frac{dv}{ds}).(\frac{ds}{du})^2$$

$$= -[\lambda + 2\mu\frac{dv}{du} + \gamma(\frac{ds}{du})^2] + [l\frac{dv}{du} + 2m(\frac{dv}{du})^2 + n(\frac{dv}{du})^3)]$$

Substitute (1) in (4)

$$\frac{d^2v}{du^2} = -\lambda - 2\mu\frac{dv}{du} - \gamma(\frac{ds}{du})^2 + l(\frac{dv}{du}) + 2m(\frac{dv}{du})^2 + n\left(\frac{dv}{du})^3\right)$$

$$= n(\frac{dv}{du})^3) + (2\gamma - m)(\frac{dv}{du})^3 + (l - 2\mu)\frac{dv}{du}) - \lambda \ldots . (5)$$

From the existence and uniqueness of solution of the initial value problem of an ODE of second order there exists a unique solution of γ of (5) with initial condition $\gamma = \gamma_0 \, and \, \frac{d\gamma}{du} = \gamma_1 \, at \; u = u_0$

Thus any solution u,v of (5) gives the direction coefficient of the tangent at P.

Hence a geodesic is uniquely determined by the initial point P and condition.

Hence the Proof.

9.4 Check your progress

- Define geodesic parallel
- Define geodesic curvature
- Define geodesic parameters
- Derive the orthogonal family of geodesic

NOTES

9.5 Summary

- A geodesic can be found to pass through any given point and have any given direction on a surface. the geodesic is uniquely determined by the initial condition.

- If α is such that variation s(α) is atmost at order ε^2 for all some variation in α for different $\lambda(t)$ and $\mu(t)$. Then $S(\alpha)$ is said to be stationary and α is geodesic.

- A necessary and sufficient condition for a curve u = u(t) and v = v(t) on a surface r = r(u,v) to be a geodesic is that,
  - $U\frac{\partial t}{\partial \dot{u}} - V\frac{\partial t}{\partial \dot{v}} = 0\ldots\ldots(1)$, where $U = \frac{d}{dt}\frac{\partial t}{\partial \dot{u}} - \frac{\partial t}{\partial u} = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{u}}, V = \frac{d}{dt}\frac{\partial T}{\partial \dot{v}} - \frac{\partial T}{\partial v} = \frac{1}{2T}.\frac{dT}{dt}.\frac{\partial T}{\partial \dot{v}}\ldots\ldots(2)$

- When the parametric curves are orthogonal.

v= constant is a geodesic iff E_2 =0

u=constant is a geodesic iff G_1=0

9.6 Keywords

Geodesic:
Let A and B be two given points on a surfaces. Let these points be joined by curves lyping on S. Then any curve possessing stationary length for small variation over S is called Geodesic.
Stationary:
If α is such that variation s(α) is atmost at order ε^2 for all some variation in α for different $\lambda(t)$ and $\mu(t)$. Then $S(\alpha)$ is said to be stationary and α is geodesic.

9.7 Self Assessment Questions and Exercises

1. Show that the curve u+v= constant are geodesic on the surface with the metric (1+u^2) $du^2 - 2uvdudv + (1 + v^2)dv^2$.
2. Show that the curves of the family u=ct^2, v=ct^3 are geodesic on the surface with the metric $2v^2du^2 - 2uvdudv + u^2dv^2$,u>0, v>0.
3. Show that if E,F,G are functions of u only and $\frac{F^2}{E}$ is constant, then the parametric curves v= constant are all geodesics.
4. Show that the geodesics on the surface of revolution x=u cos $\theta, y = u\sin\theta, z = f(u)$ are given by $u^2\frac{d\theta}{ds}$ = constant.
5. For the anchor ring r=((b+acosu)cosv, (b+acosu)sinv,z=a)

i) Verify the relation $Uu + \dot{V}\dot{v} = \frac{dT}{dt}$

ii) Obtain the two differential equations of a geodesic other than the merdians and parallels.

iii) Express $\frac{dv}{du}$ as a function of u for geodesic other than parametric curves.

9.8 Further Readings

NOTES

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

UNIT-X GEODESIC PARALLELS

NOTES

Structure

10.1 Introduction

Since geodesics on surfaces behave like straight lines in planes, we formulate a coordinate system on a surface with the help of geodesics. As a prelude to this, we introduce geodesic parallels and geodesic curvature in this chapter.

10.2 Objectives

After going through this unit, you will be able to:

- Define geodesic curve
- Derive the parametric system of geodesic parallel
- Derive the equation of orthogonal families of geodesic parallel.
- Define orthogonal trajecotories of the given family of geodesics

10.3 Geodesic Parallels.

Theorem:

For any gives family of geodesic on a surface, a parametric system can be chosen so that metric takes the form $ds^2 = du^2 + G(u,v)dv^2$. the given geodesic are the parametric curves v=constant and their orthogonal trajectories are given by u=constant, u being the distance measured along a geodesic from a fixed parallel.

Proof:

For a given family of geodesic curves.

Let us take a system of parameters such that the geodesic of the family are given by v=constat and their trajectories are given by u=constant.

Since v=constant and u=constantform an orthogonal parametric system F=0.

We know that,

v=constant is geodesic iff $EE_2 + FE_1 - 2EF_1 = 0$

Since $E \neq 0$ and F=0 the above condition reduce to $E_2 = 0$

E is independently of v and its a function of u only.

The metric becomes,

$$ds^2 = E(u)du^2 + G(u,v)dv^2 \ldots\ldots (1)$$

Now consider, the orthogonal trajector's $u = u_1 and u_2$ and find the distance between them along the geodesic v=constant.

Since v=c, dv=0

$$(1) \Rightarrow ds^2 = E(u)du^2 ds$$
$$= \sqrt{E(u)du} \ldots\ldots (2)$$
$$Integrating, (2) we get$$

NOTES

$$,S = \int_{u_1}^{u_2} (\sqrt{E(u)})du.....(3)$$

Since S is independent of v=constant the distance between orthogonal trajectories, is some along any geodesic v=constant.
Therefore, orthogonal trajectories are parallel.
Let the distance from some fixed point parallel to the neighboring parallel be du.
Then ds=du and dv=0

$$(1) \Rightarrow du^2 = E(u)du^2$$
$$E(u) = 1$$
$$(1) \Rightarrow ds^2 = du^2 + G(u,v).dv^2$$

Hence the proof.

***Geodesic parameteric*:**

Definition:

The orthogonal trajectories of the given v=constant on a surface called Geodesic Parallels family of geodesic, u and v are called geodesic parameters.

Definition:

The geodesic form $du^2 + G(u,v)dv^2$ is called the geodesic form of ds^2.

Example:

In the plane, we know that the straight lines are geodesics. Now consider a family of straight lines enveloping the given curve C. This family of straight lines envelopes C so that C becomes the evolute and these family of straight lines are normal to the involute. Hence the geodesic parallels are the involutes of C.

Example:

Let the family of geodesics be the straight lines concurrent at a point O. Then the geodesic parallels are the concentric circles with centre O. Since the concentric crcles cut the family of straight line through O orthogonally, the concentric circles form a family of orthogonal trajectories which are the geodesic parallels.

Theorem:

If a surface admits two orthogonal families of geodesic then it is isometric with the plane.

Proof:

Let U=constant be a family of geodesic.
Then the family of orthogonal trajectories is u=constant.
Suppose if we take u=constant is a family of orthogonal trajectories.
Therefore, the surface admits the two orthogonal family of geodesic v=constant.
Let the distance from some fixed parallel to the neighboring parallel to du.
Hence ds=du and dv=0

$$\Rightarrow ds = E.du \; gives \quad du = E.duE(u) = 1$$

Similarly, measuring the distance along the geodesic u=constant. Let the distance along the geodesic u=constant.
Let the distance from some fixed parallel to the neighboring parallel to dv.

$$Hence \;\; ds = dv \;\; and \;\; du = 0$$
$$ds = G(u,v)dv$$
$$dv = G(u,v)dv$$
$$G(u,v) = 1$$

Thus the metric becomes, $ds^2 = du^2 + dv^2$ which is the metric of the plane.

NOTES

10.4 Check your progress

- Define geodesic parallel
- Define geodesic parametric
- Write the geodesic form

10.5 Summary

- For any gives family of geodesic on a surface, a parametric system can be chosen so that metric takes the form $ds^2 = du^2 + G(u,v)dv^2$.
- If a surface admits two orthogonal families of geodesic then it is isometric with the plane.
- The orthogonal trajectories of the given v=constant on a surface called Geodesic Parallels family of geodesic

10.6 Keywords

Definition:
The orthogonal trajectories of the given v=constant on a surface called Geodesic Parallels family of geodesic, u and v are called geodesic parameters.
Definition:
The geodesic form $du^2 + G(u,v)dv^2$ is called the geodesic form of ds^2.

10.7 Self Assessment Questions and Exercises

1. Find the fundamental coefficients E, F, G and L, M, N for the helicoid. R(u,v)= ((u) cosv, (u) sinv, f(u)+cv). Also find the unit normal to the surface.
2. Find the position vector of a point on the surface generated by the normals of a twisted curve. Find the fundamental coefficients and the unit normal to the surface.
3. Prove that the family of geodesics on the paraboloid of revolution r=(u, $\sqrt{u}$ cosv, $\sqrt{u}$ sinv) has the form u - c^2 =u(1+4c^2) $\sin^2${v-2c log k[$2\sqrt{u-c^2}+\sqrt{4u+1}$]}, where c and k are constant.
4. Show that any curve is a geodesic o the surface generated by its binormals.
5. A particle is constrained to move on a smooth surface under no force except the normal reaction. Show that its path is geodesic.
6. Using Christoffel symbols, obtain the geodesics equation on the helicoid

r=(u cosv, u sinv, cv).

10.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

NOTES

UNIT- XI GEODESIC URVATURE

Structure

11.1 Introduction

This chapter deals with the concept of geodesic curvature. The normal curvature, geodesic curvature are also defined and some theorems are also derived.

11.2 Objectives

After going through this unit, you will be able to:

- Define geodesic curvature
- Understand the concept of geodesic curvature vector and normal curvature
- Solve the problems in geodesic curvature

11.3 Geodesic curvature

Normal curvature:

The normal component κ_n of r'' is called the normal curvature at P where $r'' = \kappa_n N + \lambda r_1 + \mu r_2$.

Geodesic curvature vector:

The vector $\lambda r_1 + \mu r_2$ with component (λ, μ) is tangential to the surface. The vector with components (λ, μ) of the tangential vector $\lambda r_1 + \mu r_2$ to the surface is called the geodesic curvature vector at P. It is denoted by K_g.

Theorem:

A curve on a surface is a geodesic iff if the geodesic curvature vector is zero.

Proof:

Let $r = r(s)$ be any curve on the surface with the principal normal n and surface normal N at P.

If (λ, μ) is the geodesic curvature vector at P, then

$$r'' = \kappa_n N + \lambda r_1 + \mu r_2$$

or

$$\kappa_n = \kappa_n N + \lambda r_1 + \mu r_2 \ldots\ldots (1)$$

Let the curve be geodesic. then by the normal property of a geodesic $n = N \ldots (2)$

Using (2) in (1), we get $\kappa N = \kappa_n N + \lambda r_1 + \mu r_2$

Equating the coefficient of $r_1 and r_2$ on both sides, we get,

$\lambda = \mu = 0$ so that $k_g = 0$

Conversely,

Let $k_g = 0 \Rightarrow \lambda = 0; \mu = 0$

Hence from (1) we get $\kappa n = \kappa_n N$

Thus, the principal normal to the curve is parallel to the surface normal.

Therefore, the curve is geodesic by the normal property.

Theorem:
The geodesic curvature vector of any curve is orthogonal to the curve.
Proof:
If (λ, μ) is the curvature vector of the curve r=r(s) at P, then by previous theorem,

$$\kappa_n = \kappa_n N + \lambda r_1 + \mu r_2 \ldots\ldots (1)$$

Since, t is tangent vector to the curve as well as to the surface $n.t = 0$ $and\ \ N.t = 0 \ldots . (2)$
Taking dot product with t on both side of (1) and using (2).
We obtain , $(\lambda r_1 + \mu r_2).t = 0$ which proves that (λ, μ) is orthogonal to the curve.
Theorem:
For any curve on a surface the geodesic curvature vector is intrinsic.
Proof:
To Prove: vector (λ, μ) is intrinsic.
We have to show that (λ, μ) can be found out from the metric of the surface.

$$r''.r_1 = U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \left(\frac{\partial T}{\partial u}\right)$$

$$r''.r_2 = V = \frac{d}{ds}(\frac{\partial T}{\partial v'}) - (\frac{\partial T}{\partial v}) = 0 \ldots . (1)$$

If (λ, μ) is the geodesic curvature, vector at a point on the surface, then $r'' = \kappa_n N + \lambda r_1 + \mu r_2 \ldots . (2)$
Taking scalar product with $r_1 and r_2$ on the both sides of (2) respectively,

$$r''.r_1 = \kappa_n N.r_1 + \lambda r_1.r_1 + \mu r_2.r_1$$

$$r''.r_2 = \kappa_n N.r_2 + \lambda r_1.r_2 + \mu r_2.r_2 \ldots .. (3)$$

Using $N.r_1 = 0 and N.r_1 = 0$ and the first fundamental coefficient, we have from (1),

$$r''.r_1 = U = Er + \mu F$$

$$r''.r_2 = V = \lambda F + \mu G \ldots . (4)$$

Solving for (λ, μ) interms of U and V from (4) we obtain,

$$\lambda = \frac{1}{H^2}(UG - HF);\ \ \mu = \frac{EV - FU}{H^2};\ \ H^2 = EG - F^2$$

which shows that the vector (λ, μ) is intrinsic.
Theorem:
The condition of orthogonality of the geodesic curvature vector (λ, μ) with any vector (u,v) on a surface is $u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0$
Proof:
The tangential direction at a point (u,v) on a surface is (u', v').
Since (λ, μ) and (u', v') are orthogonal
Using l=$\lambda, m = \mu, l' = \lambda', m' = \mu'$, in the condition of orthogonality.

$$Ell' + F(lm' + l'm) + Gmm' = 0$$

We obtain , $E\lambda u' + F(\lambda v' + u'\mu) + G(\mu v') = 0,$
which can be written as,

$$u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0$$

Since $U = E\lambda + F\mu and V = F\lambda + G\mu$, we rewrite the above condition as $Uu' + Vv' = 0$

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Theorem:

In the notation of the christoffel symbols, the components of the geodesic curvature vector are,

$$\lambda = u'' + \Gamma_{11}^{1}u'^2 + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}v'^2$$
$$\mu = v'' + \Gamma_{11}^{2}u'^2 + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}v'^2$$

Proof:

Taking $r = r(s),\ r_1 = r^1 = r_1u' + r_2v'' and$

$$hence\ r'' = r_1u'' + r_2v'' + r_{11}u'^2 + 2r_{12}u'v' + r_{22}v'^2 \ldots.(1)$$

Taking dot product with r_1 on both sides,

$r''.r_1 =$
$r_1^2u'' + r_1.r_2v'' + r_1.r_{11}u'^2 + 2r_{12}r_1u'v' + r_{22}r_1.v'^2 \ldots..(2)$, since $r''.r_1 = U = E\lambda + F\mu$

Using the fundamental coefficient in (2) we obtain,

$$E\lambda + F\mu = Eu'' + Fv'' + r_1.r_{11}u'^2 + 2r_{12}r_1u'v' + r_{22}r_1.v'^2 \ldots.(3)$$

In a similar manner,

Taking scalar product with $r_2\ on\ both\ side$ and using $r''.r_2 = V = F\lambda + G\mu$, we have

$$r''.r_1 = r_1^2u'' + r_1.r_2v'' + r_1.r_{11}u'^2 + 2r_{12}r_1u'v' + r_{22}r_1.v'^2$$
$$F\lambda + G\mu = Fu'' + Gv'' + r_2.r_{11}u'^2 + 2r_{12}r_2u'v' + r_{22}r_2.v'^2 \ldots(4)$$

Solving for λ from (3) and (4).

$$\lambda(EG - F^2) = (EG - F^2)u'' + [Gr_{11}.r_1 - Fr_{11}.r_2]u'^2$$
$$+2[Gr_{12}.r_1 - Fr_{12}.r_2]u'v' + 2[Gr_{22}.r_1 - Fr_{22}.r_2]v'^2$$

Using definition on christoffel of second kind,

$$\lambda = u'' + \Gamma_{11}^{1}u'^2 + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}v'^2$$

In a similar manner,

Solving for μ from (3) and (4)

$$\mu(EG - F^2) = (EG - F^2)v'' + [Er_{11}.r_2 - Fr_{11}.r_1]u'^2$$
$$+2[Er_{12}.r_2 - Fr_{12}.r_1]u'v' + 2[Er_{22}.r_2 - r_{22}.r_2]v'^2$$

Using the definition of the christoffel symbol of the second kind we get,

$$\mu = v'' + \Gamma_{11}^{2}u'^2 + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}v'^2$$

Theorem:

With S as parameter, the components of the geodesic curvature vector are given by,

$$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = \frac{-1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}$$
$$\mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = \frac{-1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'}$$

Proof:

From theorem (3), we have

$$\lambda = \frac{1}{H^2}(UG - VF); \mu = \frac{-1}{H^2}(EV - FU) \ldots..(1)$$

From theorem:(4), we have $Uu' + Vv' = 0$ which gives $V = \frac{-Uu\prime}{v\prime} and \frac{U}{v\prime} = \frac{-V}{u\prime} \ldots..(2)$

Using the value V in λ in (1), we get

$$\lambda = \frac{1}{H^2}(GU + F\frac{-Uu'}{v'}) = \frac{U}{H^2v'}(v'G + u'F) \ldots..(3)$$

To complete the proof, it is enough if show that:$(v'G + u'F) = \frac{\partial T}{\partial v\prime}$

Now, $T = \frac{1}{2}\left[Eu'^2 + 2Fu'v' + Gv'^2\right]$

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$$Hence,$$
$$\frac{\partial T}{\partial u'} = Eu' + Fv', \qquad \frac{\partial T}{\partial v'} = Fu' + Gv' \ldots\ldots (4)$$

Using (3) in (4) we have,

$\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = \frac{-1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}$ [since by (2)]

Substitute (2) in (1), for μ we get,

$$\mu = \frac{-1}{H^2}\left(E\frac{Uu'}{v'} + FU\right) = \frac{-1}{H^2}\frac{U}{v'}(u'E + v'F) \ldots\ldots (5)$$

Using (4) in (5), we get,

$\mu = \frac{-1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'} = \frac{V}{H^2 u'}.\frac{\partial T}{\partial u'}$ [since by (2)]

1. Obtain the geodesic curvature vector of a curve on a right helicoid $\boldsymbol{r} = (\boldsymbol{u\cos v}, \boldsymbol{u\sin v}, \boldsymbol{av})$ using different formula for it.

Solution:

We have derived three different formulae for the geodesic curvature vector . we obtain the geodesic curvature vector using these three different formulae.

Now, $r_1 = (\cos v, \sin v, 0)$

$$r_2 = (-u\sin v, u\cos v, a)$$

Hence, $E = r_1.r_1 = 1;\ \ F = 0$

$$G = a^2 + u^2;\ \ H = \sqrt{a^2 + u^2}$$

(i) $\lambda = \frac{1}{H^2}(UG - VF); \quad \mu = \frac{-1}{H^2}(EV - FU)$

Let us find U and V in the above formulae.

$$T = \frac{1}{2}\left[u'^2 + (u^2 + a^2)v'^2\right]$$
$$\frac{\partial T}{\partial u'} = u'; \quad \frac{\partial T}{\partial v'} = (u^2 + a^2)v'$$
$$\frac{\partial T}{\partial u} = uv'^2; \quad \frac{\partial T}{\partial v} = 0 \ldots\ldots (1)$$

Hence, $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = u'' - uv'^2 \ldots\ldots (2)$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = (u^2 + a^2)v'' + 2uu'v' \ldots\ldots (3)$$

Hence $\lambda = \frac{1}{a^2+u^2}\left(u'' - uv'^2\right).(a^2 + u^2)\lambda = u'' - uv'^2$

$$\mu = \frac{1}{a^2 + u^2}\left[(u^2 + a^2)v'' + 2uu'v'\right]$$

ii). $\lambda = u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2$

$$\mu = v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2$$

First let us calculate the christoffel symbols in the above formula for a curve on the right helicoid.

$$\Gamma111 = \frac{1}{2}E_1 = 0; \qquad \Gamma112 = \frac{1}{2}E_2 = 0$$
$$\Gamma122 = F_2 - \frac{1}{2}G_1 = -u; \qquad \Gamma211 = F_1 - \frac{1}{2}E_2 = 0$$
$$\Gamma212 = \frac{1}{2}G_1 = u; \qquad \Gamma222 = \frac{1}{2}G_2 = 0$$

Hence, $\Gamma 11^1 = \frac{1}{H^2}(G\Gamma111 - F\Gamma211) = 0$

$$\Gamma 12^1 = \frac{1}{H^2}(G\Gamma112 - F\Gamma212) = 0$$

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$$\Gamma 22^1 = \frac{1}{H^2}(G\Gamma 122 - F\Gamma 222) = \frac{1}{a^2+u^2}(a^2+u^2)(-u) = -u$$

$Therefore, \lambda = u'' - uv'^2$

TO find: μ

Let us find other christoffel symbol of the second kind.

$$\Gamma 11^2 = \frac{1}{H^2}(E\Gamma 211 - F\Gamma 111) = 0$$

$$\Gamma 12^2 = \frac{1}{H^2}(E\Gamma 212 - F\Gamma 112) = \frac{u}{a^2+u^2}$$

$$\Gamma 22^2 = \frac{1}{H^2}(G\Gamma 222 - F\Gamma 122) = 0$$

$$Therefore, \mu = v'' + \frac{2uu'v'}{a^2+u^2} = \frac{1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v']$$

iii). $\lambda = \frac{1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial v'} = \frac{-1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial v'}$

$$\mu = \frac{1}{H^2}\frac{V}{u'}\frac{\partial T}{\partial u'} = \frac{-1}{H^2}\frac{U}{v'}\frac{\partial T}{\partial u'}$$

Using (1), (2) and (3),

$$\lambda = \frac{1}{a^2+u^2}\left(\frac{u''-uv'^2}{v'}\right).v'(a^2+u^2) = u'' - uv'^2$$

The other expansion gives the formula for $\lambda intermsofv''asfollows,$

$$\lambda = \frac{-1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v'].\frac{v'}{u'}(a^2+u^2)$$

$$\lambda = \frac{v'}{u'}[(u^2+a^2)v'' + 2uu'v']$$

Further using (1), (2) and (2) in the formula for μ , we have,

$$\mu = \frac{1}{a^2+u^2}[(u^2+a^2)v'' + 2uu'v']$$

and the alternate expression for μ, is

$$\mu = \frac{-1}{a^2+u^2}\left(\frac{u''-uv'^2}{v'}\right).u'$$

Now we are in a position to define the geodesic curvature and derive the formula for it interms of the parameter S and t.

Definition:

The geodesic curvature at any point of a curve denoted by κ_g is defined as the magnitude of its geodesic curvature vector with proper sign. κ_g is considered to be positive or negative according as the angle between the tangent to the curve and the geodesic curvature vector is $\frac{\Pi}{2} or \frac{-\Pi}{2}$, so we have $\kappa_g = \pm\sqrt{\lambda^2+\mu^2}$

Theorem:

If $r = r(\text{S})$ is the position vector of a point P of a curve on a surface then,

i). $\kappa_g = [N, r', r'']$

$ii). \kappa_g = \dot{S}^{-3}[N, \dot{r}, \ddot{r}]$

Proof:

From theorem :2,

The geodesic curvature vector is orthogonal to the unit tangent vector $r' = \frac{dr}{ds}$ at P.

Since the geodesic curvature vector $\lambda r_1 + \mu r_2$ lies in the tangent plane at P. it is orthogonal to this surface normal N at P. Thus the geodesic curvature vector is orthogonal to both N and r' and therefore it is parallel to the unit vector $N \times r'$.

Since κ_g is the magnitude of the geodesic curvature vector,

We can take the geodesic curvature vector $\lambda r_1 + \mu r_2 \ as \ \kappa_g(N \times r') \ldots\ldots (1)$

We know that, $r'' = \kappa_n N + \lambda r_1 + \mu r_2 \ldots. (2)$

Using (1) and (2) we obtain $r'' = \kappa_n N + \kappa_g(N \times r') \ldots.. (3)$

Taking scalar product with unit vector $(N \times r')$ on both side of (3) we obtain,

$$(N \times r').r'' = (N \times r').[\kappa_n N + \kappa_g(N \times r')]$$

Since $N.(N \times r') = 0 and (N \times r').(N \times r') = 1$,we get from (4),

$\kappa_g = [N, r', r'']$ which proves (i)

ii). we shall rewrite the formula (i) by using any parameter t.

Now, $\dot{r} = \frac{dr}{dt} = \frac{dr}{ds}.\frac{ds}{dt} = r'^{\dot{S}}$

$$\ddot{r} = \frac{d}{dt}(r'\dot{S}) = \frac{d}{ds}(r'\dot{S}).\frac{ds}{dt} = r''\dot{S}^2 + r'\dot{S}$$

Since $r\dot{r} \times r'\ddot{S} = 0$ we have,

$$\dot{r} \times \ddot{r} = r'^{\dot{S}} * \left(r''^{\dot{S}^2} + r'^{\dot{S}}\right) = r' \times r''^{\dot{S}^3}$$

$$\dot{r} \times \ddot{r} = \frac{1}{\dot{S}^3}(\dot{r} \times \ddot{r})$$

Hence from the formula (i), we have

$$\kappa_g = [N, r', r'']$$

$$\kappa_g = \frac{1}{\dot{S}^3}[N, \dot{r}, \ddot{r}]$$

Corollary:

$$\kappa_g = \dot{S}^3 H^{-1}[(r_1.\dot{r})(r_2.\ddot{r}) - (r_2.\dot{r})(r_1.\ddot{r})]$$

Proof:

Since $HN = r_1 \times r_2$ we have $N = H^{-1}(r_1 \times r_2)$ using this value of N in the above lemma,

$$\kappa_g = \dot{S}^3 H^{-1}(r_1 \times r_2)(\dot{r} \times \ddot{r}) \ldots\ldots (1)$$

$Now, (r_1 \times r_2)(\dot{r} \times \ddot{r}) = (r_1.\dot{r})(r_2.\ddot{r}) - (r_2.\dot{r})(r_1.\ddot{r}) \ldots (2)$

Using (2) in (1) we obtain,

$$\kappa_g = \dot{S}^{-3} H^{-1}[(r_1.\dot{r})(r_2.\ddot{r}) - (r_2.\dot{r})(r_1.\ddot{r})]$$

As an application of the above corollary. we derive the formula for κ_g in the most simplest form interms of the instrinsic quantities of a surface U and V.

Theorem:

If the U and V are the intrinsic quantities of a surface at a point (u,v) then,

$$i): \kappa_g = \frac{1}{H}\frac{V(s)}{u'} and$$

$$ii): \kappa_g = \frac{-1}{H}\frac{U(s)}{v'}$$

Proof:

$$T = \frac{1}{2}\dot{r}^2 = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$$

Hence $\frac{\partial T}{\partial \dot{u}} = \dot{r}.\frac{\partial \dot{r}}{\partial \dot{u}}.\frac{\partial T}{\partial \dot{v}} = \dot{r}.\frac{\partial \dot{r}}{\partial \dot{v}} \ldots. (1)$

NOTES

But $\frac{dr}{dt} = \frac{\partial r}{\partial u}.\dot{u} + \frac{\partial r}{\partial v}.\dot{v} = r_1\dot{u} + r_2\dot{v}$

$$so\ that, \frac{\partial \dot{r}}{\partial \dot{u}} = r_1\ and\ \frac{\partial \dot{r}}{\partial \dot{v}} = r_2 \ldots (2)$$

Thus, from (1) and (2), $\frac{\partial T}{\partial \dot{u}} = \dot{r}.r_1$; $\frac{\partial T}{\partial \dot{v}} = \dot{r}.r_2 \ldots\ldots (3)$

We know that, $U(t) = \ddot{r}.r_1$; $V(t) = \ddot{r}.r_2 \ldots. (4)$

From theorem 1 of 3.5

Using (3) and (4) in the corollary,

$$\kappa_g = \frac{1}{H\dot{S}^3}[V(t).\frac{\partial T}{\partial \dot{u}} - U(t).\frac{\partial T}{\partial \dot{v}}]$$

If we take S as parameter in place of t in the above equation,

$$\kappa_g = \frac{1}{H}\left[V(S).\frac{\partial T}{\partial u'} - U(S).\frac{\partial T}{\partial v'}\right]\ as\ \dot{S} = 1 \ldots\ldots (5)$$

Since $u'U(S) + v'V(S) = 0$, we have $U(S) = -\frac{v'}{u'}.V(S) \ldots\ldots (6)$

Using (6) in (5)

$$\kappa_g = \frac{V(S)}{Hu'}[u'.\frac{\partial T}{\partial u'} - v'.\frac{\partial T}{\partial v'}]$$

From Euler's theorem on homogeneous function,

$u'.\frac{\partial T}{\partial u'} - v'.\frac{\partial T}{\partial v'} = 2T$ so that the above equation becomes $\kappa_g = \frac{V(S)}{Hu'}.2T \ldots\ldots\ldots (7)$

Since s is the parameter $r'^2 = 1$, so that $T = \frac{1}{2}r'^2 = \frac{1}{2}$

Using this value of T in (7)

$$\kappa_g = \frac{V(S)}{Hu'}$$

Similarly eliminating V(S) in (5), we obtain

$$k_g = -\frac{1}{H}.\frac{U(S)}{v'}$$

which completes the proof of theorem.

Corollary:

If (λ, μ) is the geodesic curvature vector of a curve then,

$$\kappa_g = \frac{-H\lambda}{Fu' + Gv'} = \frac{H\mu}{Eu' + Fv'}$$

Proof:

From theorem; 3 we have,

$$\lambda = \frac{1}{H^2}(GU - FV);\quad \mu = \frac{1}{H^2}(EV - FU) \ldots\ldots (1)$$

Since we take S as the parameter $Uu' + Vv' = 0$

So that $U = \frac{Vv'}{u'}\ and\ V = \frac{-Uu'}{v'} \ldots\ldots (2)$

Using $\lambda = \frac{1}{H^2}\frac{U}{v'}(Gv' - Fu')$; $\mu = \frac{1}{H^2}\frac{V}{u'}(Eu' - Fv') \ldots\ldots (3)$

Using the theorem in (3),

$$\lambda = -\frac{\kappa_g}{H}(Gv' - Fu');\ \mu = \frac{\kappa_g}{H}(Eu' - Fv')$$

So that , we have

$$\kappa_g = \frac{-H\lambda}{Gv' - Fu'} = \frac{H\mu}{Eu' - Fv'}$$

1. Prove that all straight lines on a surface are geodesic.

Solution

Let r=r(S) be a point on a straight line on a straight.

Then $r' = t$

$$r'' = \frac{dt}{ds} = \kappa\bar{n}$$

Since r=r(S) is a straight line $\kappa = 0$ so that $r'' = 0$

The geodesic curvature of a curve on a surface is $\kappa_g = [N, r', r'']$, $since\ \kappa = 0\ and\ r'' = 0$

Thus the geodesic curvature of a straight line on a surface is zero which implies by property (ii) after the definition of geodesic curvature all straight lines on a surface are geodesics.

Theorem:

If $\kappa_a and \kappa_b$ denotes the geodesic curvature of the parameter curve u=constant and v=constant respectively. Then,

$$\kappa_a = \frac{1}{2H} E^{\frac{-3}{2}} [2EF_1 - EE_2 - FE_1]$$

$$\kappa_b = \frac{1}{2H} G^{\frac{-3}{2}} [FG_2 + GG_1 - 2GF_2]$$

Proof:

For the parametric curve v=constant.

Let us take u itself as the parameter. so that $\dot{u} = 1 and \dot{v} = 0$

Now, $T = \frac{1}{2}[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2]$

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}E2\dot{u} + \frac{1}{2}2F\dot{v} = E\dot{u} + F\dot{v}$$

$$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}2F\dot{u} + \frac{1}{2}G.2\dot{v} = F\dot{u} + G\dot{v}$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}[E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2]$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}[E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2]$$

From our choice of parameter we have from the above equation,

$$\frac{\partial T}{\partial \dot{u}} = E; \quad \frac{\partial T}{\partial \dot{v}} = F$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}.E_1; \quad \frac{\partial T}{\partial v} = \frac{1}{2}.E_2 \ldots.(1)$$

$[since\ by\ \dot{u} = 1, \dot{v} = 0]$

Further $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \left(\frac{\partial T}{\partial u}\right)$

$$V = \frac{d}{dt}(\frac{\partial T}{\partial \dot{v}}) - (\frac{\partial T}{\partial v}) \ldots..(2)$$

Substitute (1) in (2), we have,

$$U = \frac{d}{dt}(E) - \frac{1}{2}E_1 = \frac{\partial E}{\partial u}\dot{u} + \frac{\partial E}{\partial v}\dot{v} - \frac{1}{2}E_1$$

$$= E_1\dot{u} + E_2\dot{v} - \frac{1}{2}E_1$$

$$= E_1 + 0 - \frac{1}{2}E_1 [since by \dot{u} = 1, \dot{v} = 0]$$

$$U = \frac{1}{2}E_1 \ldots..(3)$$

$$V = \frac{d}{dt}(F) - \frac{1}{2}E_2 = \frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial v}\dot{v} - \frac{1}{2}E_2$$

$$= F_1\dot{u} + F_2\dot{v} - \frac{1}{2}E_2$$

NOTES

$$= F_1 + 0 - \frac{1}{2}E_2 [since by \dot{u} = 1, \dot{v} = 0]$$

$$V = F_1 - \frac{1}{2}E_2 \ldots . (4)$$

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$ becomes $\dot{S}^2 = E$ so that $\dot{S} = \sqrt{E} \ldots . (5)$

We know that,

$\kappa_g = \frac{1}{H\dot{S}^3}[V(t).\frac{\partial T}{\partial \dot{u}} - U(t).\frac{\partial T}{\partial \dot{v}}] \ldots . (6)$ [since by above theorem equation (5)]

Substitute (1), (3),(4) and (5) in (6),

$$\kappa_a = \frac{1}{H(\sqrt{E})^3}\left[E\left(F_1 - \frac{1}{2}E_2\right) - F\left(\frac{1}{2}E_1\right)\right]$$

$$= \frac{1}{H(E)^{\frac{3}{2}}}[EF_1 - \frac{1}{2}]EE_2 - \frac{1}{2}FE_1]$$

$$= \frac{1}{H(E)^{\frac{3}{2}}}.\frac{1}{2}(2EF_1 - EE_2 - FE_1)$$

$\kappa_a = \frac{1}{2H}.(E)^{\frac{3}{2}}.(2EF_1 - EE_2 - FE_1)$

ii). Let us take v itself as the parameter, so that $\dot{v} = 1$ and $\dot{u} = 0$

Using these we have,

$$\frac{\partial T}{\partial \dot{u}} = F; \quad \frac{\partial T}{\partial \dot{v}} = G$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}.G_1; \qquad \frac{\partial T}{\partial v} = \frac{1}{2}.G_2 \ldots .. (7)$$

Substitute (7) in (2)

$$U = \frac{d}{dt}(F) - \frac{1}{2}G_1 \ = \frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial v}\dot{v} - \frac{1}{2}G_1$$

$$= F_1\dot{u} + F_2\dot{v} - \frac{1}{2}G_1$$

$$= F_1\dot{u} + F_2\dot{v} - \frac{1}{2}G_1[since by \dot{u} = 0, \dot{v} = 1]$$

$$U = F_2 - \frac{1}{2}G_1 \ldots .. (8)$$

$$V = \frac{d}{dt}(G) - \frac{1}{2}G_2$$

$$= \frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial v}\dot{v} - \frac{1}{2}G_2$$

$$= G_1\dot{u} + \frac{\partial G}{\partial v}\dot{v} - \frac{1}{2}G_2$$

$$= G_1\dot{u} + G_2\dot{v} - \frac{1}{2}G_2[since by \dot{u} = 0, \dot{v} = 1]$$

$$V = \frac{1}{2}G_2 \ldots ... (9)$$

$Since\ ds^2 = Edu^2 + 2Fdudv + Gdv^2$ becomes $\dot{S}^2 = G$

So that $\dot{S} = \sqrt{G} \ldots . (10)$

Substitute (7), (8),(9) and (10) in (6),

$$\kappa_b = \frac{1}{H(\sqrt{G})^3}\left[F\left(\frac{1}{2}G_2\right) - G\left(F_2 - \frac{1}{2}G_1\right)\right]$$

$$= \frac{1}{H(G)^{\frac{3}{2}}}[\frac{1}{2}]FG_2 - GF_2 + \frac{1}{2}GG_1]$$

$$\kappa_b = \frac{1}{2H}.(G)^{\frac{-3}{2}}.(FG_2 - 2GF_2 - GG_1)$$

NOTES

Corollary:

When the parametric are orthogonal then,

$i).\kappa_a = \frac{-1}{\sqrt{EG}}.\frac{\partial}{\partial v}\sqrt{E}$ $\quad ii).\kappa_b = \frac{1}{\sqrt{EG}}.\frac{\partial}{\partial u}\sqrt{G}$

Proof:

Since the parametric curves are orthogonal, F=0,

i). From above theorem,

$$\kappa_a = \frac{1}{2H}.(E)^{\frac{3}{2}}.(2EF_1 - EE_2 - FE_1)$$

$$= \frac{-EE_2}{2\sqrt{EG}.E^{\frac{3}{2}}}[since by F = 0, H = \sqrt{EG - F^2}]$$

$$= \frac{-EE_2}{2\sqrt{EG}.E^{\frac{3}{2}-1}}$$

$$= \frac{-E_2}{2\sqrt{EG}\sqrt{E}}$$

$$= \frac{-1}{\sqrt{EG}}.\frac{E_2}{2\sqrt{E}}$$

$$\kappa_a = -\frac{1}{\sqrt{EG}}.\frac{\partial}{\partial v}\sqrt{E}$$

ii). From above theorem,

$$\kappa_b = \frac{1}{2H}.(G)^{\frac{-3}{2}}.(FG_2 - 2GF_2 - GG_1)$$

$$= \frac{1}{2H}.\frac{GG\prime}{G^{\frac{3}{2}}} = \frac{1}{2H}.\frac{G\prime}{G^{\frac{3}{2}-1}} = \frac{1}{2H}.\frac{G_1}{\sqrt{G}}$$

$$= \frac{1}{2\sqrt{EG}}.\frac{G_1}{\sqrt{G}} = \frac{1}{\sqrt{EG}}.\frac{1}{2}.\frac{G_1}{\sqrt{G}}$$

$$\kappa_b = \frac{1}{\sqrt{EG}}.\frac{\partial}{\partial u}\sqrt{G}$$

1. Find the geodesic curvature of the curve U=constant on the surface given by $\boldsymbol{r = (u\cos v, u\sin v, \frac{1}{2}au^2)}$

Solution:

Now, $r_1 = (\cos v, \sin v, au)$, $r_2 = (-u\sin v, u\cos v, 0)$

Hence $E = r_1.r_2 = 1 + a^2u^2$, F=0, G= $r_1.r_2 = u^2$ and H= $u\sqrt{1 + a^2u^2}$

$E_1 = 2a^2u^2$, $E_2 = 0$, $G_1 = 2u$

Using the above values in the formula for κ_b, we have,

$\kappa_b = \frac{1}{\sqrt{EG}}.\frac{\partial\sqrt{G}}{\partial u}$, $\kappa_b = \frac{1}{u\sqrt{1+a^2u^2}}$

Theorem: Liouvillie's Theorem

If θ is the angle which the curve C makes with the parametric curve v=constatn . then $\kappa_g = \theta' + Pu' + Qv'$, where $P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$

$$Q = \frac{1}{2HE}(EG_1 - FE_2)$$

Proof:

We makes use of the formula $\kappa_g = \frac{-U(s)}{HV\prime}$ in the proof.

The direction coefficient of the curve at (u,v) and the curve v=constant are respectively. (u', v') and $(\frac{1}{\sqrt{E}}, 0)$

NOTES

So if θ is the angle between the two direction (u', v') and $(\frac{1}{\sqrt{E}}, 0)$, we have from the formula, $\cos\theta = Ell' + F(lm' + ml') + Gmm'$, $\sin\theta = H(lm' - ml')$

$$\cos\theta = \frac{1}{\sqrt{E}}(Eu' + Fv') \ldots..(1), \sin\theta = \frac{HV\prime}{\sqrt{E}}$$

Now, $T = \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2]$, $\frac{\partial T}{\partial u\prime} = Eu' + Fv'$

$$\frac{\partial T}{\partial u} = \frac{1}{2}[E_1u'^2 + 2F_1u'v' + G_1v'^2] \ldots..(2)$$

Using (2) in (1), we obtain,

$$\cos\theta = \frac{1}{\sqrt{E}}.\frac{\partial T}{\partial u'} \ldots..(3)$$

Differential equation (3) w.r.t S,

$$-\sin\theta.\Theta' = \frac{1}{\sqrt{E}}.\frac{d}{ds}(\frac{\partial T}{\partial u'}) - \frac{1}{2}E^{\frac{-3}{2}}.\frac{dE}{ds}(\frac{\partial T}{\partial u'}) \ldots..(4)$$

But, $\frac{dE}{ds} = \frac{du}{ds}(\frac{\partial E}{\partial u}) + \frac{dv}{ds}(\frac{\partial E}{\partial v}) = E_1u' + E_2v' \ldots.(5)$

Using (5) in (4) , rewritting it as,

$$-\sqrt{E}.\sin\theta.\theta' = \frac{d}{ds}(\frac{\partial T}{\partial u'}) - \frac{1}{2}.\frac{1}{E}(E_1u' + E_2v').(\frac{\partial T}{\partial u'}) \ldots...(6)$$

Let us substitute for,

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) \; from \;\; U = \frac{d}{ds}(\frac{\partial T}{\partial u'}) - \frac{\partial T}{\partial u}$$

Then $-E.\sin\theta.\Theta' = U + \frac{\partial T}{\partial u} - \frac{1}{2}.\frac{1}{E}(E_1u' + E_2v').\left(\frac{\partial T}{\partial u'}\right)$

$$Using \;\; the \;\; value \;\; of \;\; \sin\theta \;\; in \;\; (4)$$

$$-Hv'\theta' = U + \frac{\partial T}{\partial u} - \frac{1}{2}.\frac{1}{E}(E_1u' + E_2v').(\frac{\partial T}{\partial u'})$$

Using (2) in the above equation , we have,

$$-Hv'\theta' = U + \frac{1}{2}[Eu'^2 + 2Fu'v' + Gv'^2] - \frac{1}{2}.\frac{1}{E}(E_1u' + E_2v').(Eu' + Fv')$$

$$= U + \frac{1}{2E}[u'v'(2EF_1 - FE_1 - EE_2) + v'^2(EG_1 - FE_2)] \ldots.(7)$$

Taking $P = \frac{(2EF_1 - FE_1 - EE_2)}{2HE}, Q = \frac{1}{2HE}(EG_1 - FE_2)$

We have from (7), $-\theta' = \frac{-U}{HV'} + u'P + v'Q \ldots..(8)$

$since \kappa_g = \frac{-U}{HV\prime}$ (8) becomes,

$$-\theta' = -\kappa_g + Pu' + Qv'$$

So that, we have $\kappa_g = \theta' + Pu' + Qv'$ which completes the proof of liouville's theorem.

Example:

If the orthogonal trajectories of the curve v=constant are geodesic prove that $(\frac{H^2}{E})$ is independent of v.

Proof:

If c is the orthogonal trajectories of the parametric curve v=constatn , we can take $\theta = \frac{\Pi}{2}$ in the theorem.

Since c is the given to a geodesic $\kappa_g = 0$.

Using these the liouville's formula becomes, $Pu' + Qv' = 0 \ldots..(1)$

Now, we shall find P and Q in the new situation.

If (u', v') and $(\frac{1}{\sqrt{E}}, 0)$ are the direction coefficient of the curve c and v=constant and $\theta = \frac{\Pi}{2}$, we have from the formula,

$\cos\theta = Ell' + F(lm' + ml') + Gmm' = Eu' + Fv' = 0 \ldots\ldots (2)$

Eliminating $u' and v'$between (1) and (2),

$$EQ - FP = 0 \ldots.. (8)$$

Substitute for P and Q in (3)

$$E(EG_1 - FE_2) - F(2EF_1 - FE_1 - EE_2) = 0$$

Solving for G from the above equation ,

$$G_1 = \frac{2EF_1 - F2E_1}{E^2} = \frac{\partial}{\partial u}.(\frac{F^2}{E})$$

So that we have $G_1 - \frac{\partial}{\partial u}.\left(\frac{F^2}{E}\right) = 0$

$$(or)\frac{\partial}{\partial u}(G_1 - (\frac{F^2}{E})) = 0$$

Nothing $G_1 - (\frac{F^2}{E}) = \frac{H^2}{E}$

We find $\frac{\partial}{\partial u}(\frac{H^2}{E}) = 0$ which implies that $\frac{P}{E}$ is independent of u.

Simply connected:

A region R of surface is said to be simply connected if every closed curve lying in the region R can be contracted or shrunk continuously into a point without learning R.

Described in a a positive sense:

A closed curve on a surface is said to described in a positive sense, if the sense of description of the curve is always left. this is nothing, but the positive rotation of $\frac{\Pi}{2}$ from the tangent to get the normal which point towards the interior of the region.

Definition:

Let R(u,v) be the given surface of class, and k be a simply connected region whose boundary is a closed curve of a class 2. Let c contains of n arcs $A_0A_1, A_1A_2, \ldots.., A_{n-1}A_n$ whose n is finite.

Theorem: Gauss Bonet theorem:

For any curve c which encloses a simply connected region R on a surface e_xC is equal to the total curvature of the R.

Proof:

We shall use liouville's formula for the $\kappa_g$and find $\int \kappa_g . ds$ with the help of Green's theorem in the for a simply connected region R bounded by C.

Lemma:

If R is a simply connected region bounded by a closed curve C. then $\int_c Pdu + Qdv = \int \int_R (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})dudv$ where P and Q are differential function of u and v in R.

Proof:

From the liouville's formula,

$$\kappa_g = \theta' + Pu' + Qv'$$

Integrating along the curve c, we have

$$\int_c \kappa_g . ds = \int_c (\theta' + Pu' + Qv')ds \ldots\ldots (1)$$

NOTES

where θ is angle between the curve c and the parametric curve v=constant and P and Q are differential function of u,v. Hence when we describe the curve C, the tangent at various members of the family v=constant described in the positive sense returns to the starting point after increasing the angle of rotation by 2Π.
This increase 2Π after complete rotation in the positive sense also includes the angle between the tangent at the finite number of vertices.
Hence we have , $\int_c d\theta + \Sigma_{r=1}^{n}\alpha_r = 2\Pi$
From the definition,

$$e_x C = 2\Pi - \Sigma_{r=1}^{n} - \int_c \kappa_g . ds \ldots\ldots (3)$$

Using (1) and (2) in (3), obtain

$$e_x C = 2\Pi - [2\Pi - \int_c d\theta] - [\int_c d\theta + Pdu + Qdv]$$

Thus, $e_x C = -\int_c Pdu + Qdv \ldots\ldots (4)$
Since R is simply connected region and p and Q are different function of u,v,
We have by Green's theorem,

$$\int_c (Pdu + Qdv) = \int \int_R (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})dudv \ldots (5)$$

Since the surface element ds= H dudv, we rewrite (5) as,

$$\int_c (Pdu + Qdv) = \frac{1}{H}\int \int_R (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})ds \ldots (6)$$

Using (6) in (4), we get,

$$e_x C = \frac{-1}{H}\int \int_R (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})d\theta \ldots (7)$$

If we take $k = \frac{-1}{H}(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})$
We can rewrite (r) as ,

$$e_x C = \int \int_R K ds \ldots\ldots (8)$$

where K is the function of u and v and it is independent of c and defined over the region R of the surface.
Next we shall show that the $e_x C$ is uniquely determined by k. If k is not unique.
Let $\overline{K}$ be such that $e_x C = \int \int_R \overline{K} ds \ldots (9)$
Using (8) and (9), we have

$\int \int_R (\overline{K} - K)ds = 0 \ldots (10)$ for every region R.

Now, let $\overline{k} \neq k$ at some point A of R. Then we must have $\overline{k} > K \; or \; \overline{k} < k$ at A.
Let us first consider $\overline{k} > K$. since the given surface is of class 3. $\frac{\partial Q}{\partial u} and \frac{\partial P}{\partial v}$ are continuous in R.
So that these exists a small region R_1 of R containing the point A such that $\overline{k} - k > 0$ at every point of R_1.
For this region R containing $R_1, \int \int_R (\overline{K} - K)ds > 0$ which contradicts (10), we get similar contradiction $\int \int_R (\overline{K} - K)ds < 0$ at A where $\overline{k} < k$.
These contradiction prove that $\overline{k} = K$ at every point of R. (ie)K is uniquely determined as a function of u and v.

Defining $\int_R Kds$ as the total curvature of R we have proved that the total curvature is exactly $e_x C$ in any region R enclosed by c.
This completes the proof of Gauss Bonnet theorem.

Gaussian Curvature:

The invariant k as defined above is called the Gaussian curvature of the surface and $\int_R Kds$ is called the total curvature of integral curvature of R where R is any region whether simply connected or not.

1. Find the curvature of a geodesic triangle ABC enclosing a region R on the surface.

Solution:

Since $e_x C$ gives the total curvature of a region bounded by c.
It is enough , if we found $e_x C$ in the example where c is known ABC is geodesic triangle enclosing a region R on the surface with interior angles A, B,C so in the case the curve is the geodesic triangle $ABC = \frac{-1}{2g}\frac{\partial}{\partial u}(\frac{2gg_1}{g}) = \frac{-g_{11}}{g}$
Hence, the gaussian curvature $K = \frac{-g_{11}}{g}$
In other words, K satisfies the differential equation $g_{11} + \kappa_g$ which proves the theorem.
The origin of a geodesic polar coordinates system is artificial singularity. since the singularity is artificial the Gaussian curvature is exists there so we shall find the gaussian curvature k_0 at the origin.

Theorem:

If P is a given point on a surface and A is the area of geodesic triangle ABC containing P, then Gaussian curvature k at P is $K = \frac{A+B+C-\Pi}{\Delta}$ where the limit is taken as the vertices $A, B, C \to P$

Proof:

From example 1:
For a geodesic triangle ABC with interior angle A, B,C on a surface the total curvature is $A + B + C - \Pi$.
Hence we have $\int_\Delta \mathrm{Kdx} = A + B + C - \Pi$ where k is a constant.

$$\int_\Delta kds = k\Delta = A + B + C - \Pi$$

$$\Rightarrow k = \frac{A + B + C - \Pi}{\Delta}$$

Hence the Gaussian curvature at the point P is $k = \lim \frac{A+B+C-\Pi}{\Delta}$ where the limit is taken as the vertices tend to P.

2. Find the Gaussian curvature of every point of a sphere of radius a.

Solution:

On a sphere of radius a, the geodesic triangle is spherical triangle formed by great circle the area A of a spherical triangle is $\Delta = a^2(A + B + C - \Pi)$. Hence using theorem 1.
The Gaussian curvature at a point P on a the sphere is

$$k = \lim \frac{A + B + C - \Pi}{a^2 A + B + C - \Pi} = \frac{1}{a^2}$$

Thus the gaussian curvature of every point of the sphere is constant and it is equal to $\frac{1}{a^2}$.
The total curvature of the sphere is

NOTES

$$\int_s kdx = \frac{1}{a^2}.\int_s ds = \frac{1}{a^2}.4\Pi a^2 = 4\Pi$$

Theorem:

If E, F and G are the fundamental coefficient of a surface then,

$$\kappa = \frac{1}{H}.\frac{\partial}{\partial u}(\frac{FE_2 - EG_1}{2HE}) + \frac{1}{H}.\frac{\partial}{\partial v}(\frac{2EF_1 - FE_1 - EF_2}{2HE})$$

Proof:

From the definition of κ, we have

$$\kappa = \frac{-1}{H}(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})$$

where P and Q are given by the liouville's formula $\kappa s = \theta' + Pu' + Qv'$

Substitute for P and Q in κ

$$\kappa = \frac{-1}{H}.\frac{\partial}{\partial u}(\frac{EG_1 - FE_2}{2HE}) + \frac{1}{H}.\frac{\partial}{\partial v}(\frac{2EF_1 - FE_1 - EF_2}{2HE})$$

or we have

$$\kappa = \frac{1}{H}.\frac{\partial}{\partial u}(\frac{FE_2 - EG_1}{2HE}) + \frac{1}{H}.\frac{\partial}{\partial v}(\frac{2EF_1 - FE_1 - EF_2}{2HE})$$

where the parametric curves are orthogonal.

Hence the formula for κ assumes the formula,

$$\kappa = \frac{1}{2\Pi}\frac{\partial}{\partial u}.(\frac{G_1}{H}) + \frac{\partial}{\partial v}(\frac{E_2}{H}).$$

3. Find the Gaussian at any point of a curve sphere with representation $\boldsymbol{r = a(\sin u.\cos v, \sin u.\sin v, \cos u)}$ *where* $\boldsymbol{0 < u < 2\Pi}$ *and* $\boldsymbol{0 \le v \le 2\Pi}$.

Solution:

We know that for the sphere $E = a^2, F = 0, G = a^2\sin^2 u, H = a^2 \sin u$ *and*

$$E_2 = 0, G_1 = 2a^2.\sin u.\cos u.$$

Using this formula in the above theorem when $0 < u < 2\Pi$,

$$\kappa = \frac{-1}{2a^2\sin^2 u}.\frac{\partial}{\partial u}(\frac{2a^2.\sin u.\cos u}{a^2\sin^2 u}) = \frac{-1}{2a^2\sin^2 u}.\frac{\partial}{\partial u}.(2\cos u)$$

$$\kappa = \frac{1}{a^2}$$

4. Find the Gaussian curvature at a point (u,v) of the anchor ring $\boldsymbol{r = \big((b + a\cos u).\cos v, (b + a\cos u).\sin v, a\sin u\big)}$ *where* $\boldsymbol{0 < u, v < 2\Pi}$

Solution:

For the anchor ring, we have $E = a^2, F = 0, H = a(b + a\cos u), G = (b + a\cos u)^2$.

Since the parametric curve are orthogonal . we make use of the second formula for κ.

$$\kappa = \frac{1}{2H}[\frac{\partial}{\partial u}.(\frac{G_1}{H}) + \frac{\partial}{\partial v}(\frac{E_2}{H})]$$

Now, $E_2 = 0, G_1 = -2(b + a\cos u)a\sin u$

Hence $\kappa = \frac{1}{2a(b+a\cos u)}.\frac{\partial}{\partial u}(\frac{2a(b+a\cos u).\sin v}{a(b+a\cos u)}) = \frac{\cos u}{a(b+a\cos u)}$

which gives the gaussian curvature at any point P on the surface.

Hence the total curvature is,

$$\int\int_s Kds = \int_0^{2\Pi}\int_0^{2\Pi} KHdudv$$

$$= \int_0^{2\Pi}\int_0^{2\Pi} \frac{\cos u\, a(b + a\cos u)}{a(b + a\cos u)}.dudv$$

$= \int_0^{2\Pi} dv \int_0^{2\Pi} \cos u\, du = 2\Pi(\sin u)_0^{2\Pi} = 0$

Hence the total curvature of the whole surface is zero.

Using the geodesic polar coordinates the metric of a surface reduce to the form $ds^2 = du^2 + g^2dv^2\ where\ g^2 = G(u,v)$

Theorem:

In the geodesic polar form, the gaussian curvature $\kappa = -\frac{g_{11}}{g}\ where\ g = \sqrt{G}$ of the surface.

Proof:

Using geodesic polar co-ordinate the metric is $ds^2 = du^2 + g^2dv^2$

Hence $E = 1, F = 0, G = g^2, H = \sqrt{G} = g\ and\ G_1 = 2gg_1$

Since F=0 for formula for the gaussian curvature is

$\kappa = \frac{1}{2H}\{\frac{\partial}{\partial u}.(\frac{G_1}{H}) + \frac{\partial}{\partial v}(\frac{E_2}{H})\} = \frac{1}{2H}\frac{\partial}{\partial u}.\frac{2gg_1}{g}) = \frac{-g_{11}}{g}$.

Hence the gaussian curvature $\kappa = -\frac{g_{11}}{g}$ in other words κ satisfies the differential equation $g_{11} + \kappa_g = 0$

(The origin of a geodesic polar coordinates system is an artificial singularity.)

Theorem:

If κ_0 is the Gaussian curvature at the origin of a geodesic polar coordinates system then $g(u,0) \sim u - \frac{k_0u^3}{6} as\ u \to 0$

Proof:

Since the metric in the geodesic polar coordinates approximates to the polar form in the plane.

We can take $G \sim u^2, g \sim u\ \ and\ \ \lim_{u\to 0}\frac{\sqrt{G}}{u} = 1.\ldots.(1)$

Using theorem (3) we have $g_{11} = -k_0g \sim -k_0u.\ldots(2)$

Integrating (2)

$$g_1 \sim c_1 - k_0.\frac{u^2}{2}$$

Since $g \sim u, g_1 = 1$ so that $c_1 \sim 1$ as $u \to 0$.

Thus we have $g_1 \sim 1 - \frac{k_0u^2}{2}\ldots.(3)$

Integrating (3), once again we have,

$$g_1 \sim u - \frac{k_0u^3}{6} + c^2$$

Since $g \sim u$ which is zero at the origin $c_2 = 0$.

Hence we have $g_1 \sim u - \frac{k_0u^3}{6}$

Theorem:

If r is the radius of the geodesic circle with center at P then,

i). $(\kappa)_p = \lim_{r\to 0}.\frac{2\Pi r - c}{\frac{1}{3}.\Pi r^3}$ where c is the circumference of the geodesic circle.

ii). $(\kappa)_p = \lim_{r\to 0}.\frac{\Pi r^2 - A}{y_2.\Pi r^4}$ where A is the area of the geodesic disc.

Proof:

If P is the centre and r is the radius. then $u = r$ is the geodesic circle in geodesic polar coordinates.

NOTES

dv is the infinittesimal directed angle at P when we use the geodesic polar coordinates.
The metric of the surface becomes $ds^2 = du^2 + g^2dv^2$.
Since $u = r, du = 0$ $and\ then\ ds^2 = g^2dv^2\ \ so\ \ that\ \ ds = gdv.$
If c is the circumference of the geodesic circle then,

$$C = \int_0^{2\Pi} ds = \int_0^{2\Pi} g(r,v)dv \sim \int_0^{2\Pi} (r - \frac{k_0u^3}{6})dv$$

Hence $c \sim 2\Pi(r - \frac{k_0u^3}{6}) \ldots.(1)$[by theorem 1]
From the above formula,

$$C - 2\Pi r \sim -\frac{1}{3}\Pi k_0 r^3 \ldots.(2)$$

where k_0 is the Gaussian curvature at the origin at P.
From (2), we have $(\kappa)_p = \lim_{r\to 0}.\frac{2\Pi r - c}{\frac{1}{3}.\Pi r^3}$
ii). The elementary area $Hdudv = \sqrt{G}dudv$ where u varies from 0 to r and v varies from 0 to 2Π.
Hence the area of the geodesic disc $A = \int_0^r \int_0^{2\Pi} \sqrt{G}duds$ [by theorem 1]

$$\sqrt{G} = u - \frac{k_0u^3}{6}A = \int_0^r \int_0^{2\Pi} u - \frac{k_0u^3}{6}dudv = \int_0^r (2\Pi u - \frac{\Pi}{3}.k_0u^3)du$$

$$= \Pi r^2 - \frac{\Pi}{12}k_0 r^4$$

So solving k_0, we have

$$(\kappa)_p = \lim_{r\to 0}.\frac{\Pi r^2 - A}{y_2.\Pi r^4}$$

11.4 Check your progress

- Define geodesic curvature
- Derive the equation of geodesic curvature vector and normal
- Define gaussian curvature
- State gauss bonnet theorem

11.5 Summary

- The geodesic curvature vector of any curve is orthogonal to the curve.
- For any curve on a surface the geodesic curvature vector is intrinsic.
- The condition of orthogonality of the geodesic curvature vector (λ, μ) with any vector (u,v) on a surface is $u'(E\lambda + F\mu) + v'(F\lambda + G\mu) = 0$
- If θ is the angle which the curve C makes with the parametric curve v=constatn . then $\kappa_g = \theta' + Pu' + Qv'$where $P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$

$$Q = \frac{1}{2HE}(EG_1 - FE_2)$$

- For any curve c which encloses a simply connected region R on a surface e_xC is equal to the total curvature of the R.

NOTES

11.6 Keywords

Normal curvature:

The normal component κ_n of r'' is called the normal curvature at P where $r'' = \kappa_n N + \lambda r_1 + \mu r_2$.

Geodesic curvature vector:

The vector $\lambda r_1 + \mu r_2$ with component (λ, μ) is tangential to the surface. The vector with components (λ, μ) of the tangential vector $\lambda r_1 + \mu r_2$ to the surface is called the geodesic curvature vector at P. It is denoted by K_g.

Definition:

The geodesic curvature at any point of a curve denoted by κ_g is defined as the magnitude of its geodesic curvature vector with proper sign. κ_g is considered to be positive or negative according as the angle between the tangent to the curve and the geodesic curvature vector is $\frac{\Pi}{2} or \frac{-\Pi}{2}$, so we have $\kappa_g = \pm\sqrt{\lambda^2 + \mu^2}$

Liouvillie's Theorem:

If θ is the angle which the curve C makes with the parametric curve v=constatn . then $\kappa_g = \theta' + Pu' + Qv'$ where $P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$

$$Q = \frac{1}{2HE}(EG_1 - FE_2)$$

Gauss Bonet theorem:

For any curve c which encloses a simply connected region R on a surface $e_x C$ is equal to the total curvature of the R.

11.7 Self Assessment Questions and Exercises

1. Prove that on a surface with the metric $\mathrm{ds}^2 = \lambda^2\mathrm{du}^2 + \mu^2\mathrm{dv}^2$ the geodesic curvature of the curve u=constant is $(\lambda\mu)^{-1}\frac{\partial\mu}{\partial u}$.
2. If the parametric curves u=constant and v=constant are orthogonal, show that the geodesic curvature are $-\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}(\log\sqrt{E}), \frac{1}{\sqrt{E}}\frac{\partial}{\partial u}(\log\sqrt{G})$.
3. On the surface with $\mathrm{ds}^2 = \mathrm{e}^2\mathrm{du}^2 + \mathrm{g}^2\mathrm{dv}^2$ where e and g are functions of u, v, show that the geodesic curvature of a curve which cuts the curve v=constant at an angle θ is $k_g=\frac{d\theta}{ds} + \frac{1}{eg}(g_1 \sin\theta - e_2 cos\theta)$.
4. Find the total curvature of a pentagon enclosing a simply connected region R on a surface.
5. Find the Gaussian curvature K of a geodesic equilateral triangle on a surface.
6. Find the Gaussian curvature at a point on a cone and on a cylinder.
7. Show that the two surfaces r=(u cosv, u sinv, log u) and r=(u cosv, u sinv, v) have the same Gaussian curvture $-\frac{1}{(1+u^2)^2}$ at a corresponding points.
8. Find the isothermic system corrsponding to the analytic function f(u,v)=e^u(cos v+i sin v).

NOTES

11.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010)

BLOCK IV: LINES OF CURVATURE AND DEVELOPABLES

UNT XII THE SECOND FUNDAMENTAL FORM

Structure

12.1 Introduction

In this chapter, we studied the properties of surface in terms of the associated metrics involving E,F and G. What we did was to introduce ds^2 and to study the properties in relation to the metric which is embedded in the surface. All our study at each stage reflected the properties of the first fundamental form which are called the intrinsic properties of the surface. In this chapter we shall study the second derivative r'' of a point on a surface giving quantities pertaining to the Euclidean space in which the surface is located leading to non-intrinsic properties of a surface. The properties of surface involving normal component of vectors associated with the surface are called non-intrinsic properties.

After finding a formula for normal curvature k_n in terms of a quadratic form called the second fundamental form, we classify different points on a surface and find maximum and minimum curvature along a given direction leading to the definition of Gaussian curvature, mean curvature and principal directions. With the help of principal directions, we introduce special class of curves on a surface called lines of curvature which are characterised by Rodrique's formula. Using principal direction and normal at a point on a surface, we define Dupin's indicatrix giving rise to the definitions of conjugate and asymptotic directions on a surface. Then the envelope of the single parameter family of planes results in many developable surface whose properties we study with help of lines of curvature and Gaussian curvature. We conclude this chapter, with a study of minimal surfaces, ruled surfaces and third fundamental form.

12.2 Objectives

After going through this unit, you will be able to:

- Define envelope
- Derive the second fundamental form
- Derive Dupin's theorem
- Define lines of curvature
- Solve the problems in second fundamental form

NOTES

12.3 The second fundamental form:

Just as the moving triad (t,n,b) serves as the coordinate system at any point on a space curve, the three linearly independent vectors (N, r_1, r_2) at any point P on a surface form a coordinate system at P so that any vector through P can be represented as a linear combination of (N, r_1, r_2).

Theorem : *If k_n is the normal curvature of a curve at a point on a surface, then*

$k_n = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$ where L=N.r $_{11}$, M=N.r $_{12}$, N=N.r $_{22}$ and E,F,G are the first fundamental coefficients.

Proof:

Let r be the position vector of any point on the curve. If k_n is the normal curvature of the curve at P on a surface, we know that

$r'' = k_n N + \lambda r_1 + \mu r_2$ .....(1)

Since $r_1 . N = 0, r_2 . N = 0$, taking dot product with N on both sides of (1), we obtain

$k_n = r''. N$ ....(2)

Since r is a function of u and v, we have

$r' = \frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds} = r_1 u' + r_2 v'$ ....(3)

Diff w.r to s,

$r'' = r_1 u'' + r_2 v'' + (r_1)'u' + (r_2)'v'$ ....(4)

But $(r_1)' = \frac{\partial r_1}{\partial u} u' + \frac{\partial r_1}{\partial v} v' = r_{11} u' + r_{12} v'$ ...(5)

$(r_2)' = \frac{\partial r_2}{\partial u} u' + \frac{\partial r_2}{\partial v} v' = r_{21} u' + r_{22} v'$ ...(6)

Using (5) and (6) in (4), we get

$r'' = r_1 u'' + r_2 v'' + r_{11} u'^2 + r_{22} v'^2 + 2 r_{12} u'v'$ ....(7)

Taking dot product with N on both sides of (7) and using $r_1 . N = 0, r_2 . N = 0$ and (2), we have

$N.r'' = k_n = (N.r_{11})u'^2 + 2(N.r_{12})u'v' + (N.r_{22})v'$ ....(8)

If L=N.r $_{11}$, M=N.r $_{12}$, N=N.r $_{22}$, we have

$k_n = Lu'^2 + 2Mu'v' + Nv'$ $= \frac{Ldu^2+2Mdudv+Ndv^2}{ds^2}$

Since $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$k_n = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$ which gives the formula for the normal curvature of a curve at a point P on a surface.

Definition:

The quadratic form $Ldu^2 + 2Mdudv + Ndv^2$ is called the second fundamental form of the surface and L,M,N which are functions of u,v are called second fundamental coefficients.

Theorem *i)* $L = -N_1 . r_1, M = -N_1 . r_2 = -N_2 . r_{21}, N = -N_2 . r_2$

ii) $L = \frac{[r_{11}, r_1, r_2]}{H}, M = \frac{[r_{12}, r_1, r_2]}{H} = \frac{[r_{21}, r_1, r_2]}{H}, N = \frac{[r_{22}, r_1, r_2]}{H}$

Proof:

At any point P on the surface $r_1 and r_2$ are tangential to the surface so that we have $N. r_1 = 0$ ....(1)

and in a similar manner $N. r_2 = 0$ ....(2)

Diff.(1) w.r to u,

$N. r_{11} + N_1 . r_1 = 0$ giving $r_{11} . N = -r_1 . N_1$, since L= $N. r_{11}$ we get L=$-N_1 . r_1$

Diff.(1) w.r to v,

$N.r_{12} + N_2.r_1 = 0$ giving $r_{12}.N = -r_1.N_2$, since M= $N.r_{12}$ we get M=$-N_2.r_1$

Diff.(2) w.r to u,

$N.r_{21} + N_1.r_2 = 0$ giving $r_{21}.N = -r_2.N_1$

So that we have M=$-N_1.r_2$

Diff.(2) w.r to v,

$N.r_{22} + N_2.r_2 = 0$ giving $r_{22}.N = -r_2.N_2$, since N= $N.r_{22}$ we get N=$-N_2.r_2$

ii) To prove these formulae we use $r_1 \times r_2 = HN$ in theorem:1 for L,M,N. Now $[r_{11}, r_1, r_2] = r_{11}.(r_1 \times r_2) = r_{11}.HN = H(N.r_{11}) = HL$

Thus, $[r_{11}, r_1, r_2] = HL$ so that L=$\frac{1}{H}[r_{11}, r_1, r_2]$

Further $[r_{12}, r_1, r_2] = r_{12}.(r_1 \times r_2) = r_{12}.HN = H(N.r_{12}) = HM$

Hence we have M=$\frac{1}{H}[r_{12}, r_1, r_2] = \frac{1}{H}[r_{21}, r_1, r_2]$

Now $[r_{22}, r_1, r_2] = r_{22}.(r_1 \times r_2) = r_{22}.HN = H(N.r_{22}) = HN$ which gives N=$\frac{1}{H}[r_{22}, r_1, r_2]$

Example:

Find L,M,N for the sphere r=(acosucosv, acosusinv, asinu) where u is the latitude and v is the longitude.

r_1=(-asinu cosv, -asinu sinv, acosu)

E=a^2

r_2=(-acosusinv , -acosu cosv, 0)

G=a^2cos^2u

F=$r_1.r_2$=0,

H=$\sqrt{EG - F^2} = a^2cosu$

HN=$r_1 \times r_2 = (-a^2cos^2ucosv, -a^2cos^2usinv, -a^2sinucosu)$

Hence N=$\frac{1}{H}r_1 \times r_2$=(-cosu cosv, -cosu sinv, -sinu) which shows that the normal is directed inside the sphere.

N_1=(sinu cosv, sinu sinv, -cosu)

N_2=(cosusinv , -cosu cosv, 0)

Thus L=$-N_1.r_1 = a$, M=0 and N=$-N_2.r_2 = acos^2u$

Example:

Find the second fundamental form for the general surface of revolution.

r=(g(u)cosv, g(u)sinv,f(u)) ....(1)

From (1) $r_1 = (g_1(u)cosv, g_1(u)sinv, f_1(u))$

E=$g_1^2 + f_1^2$

$$r_2 = (-g(u)sinv, g(u)cosv, 0)$$

G=g^2

F=$r_1.r_2$=0,

H=$\sqrt{(g_1^2 + f_1^2)g}$

$$r_1 \times r_2 = (-gf_1cosv, -f_1gsinv, gg_1)$$

$$r_{11} = (g_{11}(u)cosv, g_{11}(u)sinv, f_{11}(u))$$

$$r_{22} = (-g(u)cosv, -g(u)sinv, 0)$$

$$r_{21} = (-g_1(u)sinv, g_1(u)cosv, 0)$$

HL=$r_{11}.(r_1 \times r_2) = g(g_1(u)f_{11}(u) - f_1(u)g_{11}(u))$

HM=$r_{21}.(r_1 \times r_2) = 0$

NOTES

HN=$r_{22}.(r_1 \times r_2) = f_1(u)g^2(u)$

Hence II=$\frac{1}{\sqrt{(g_1^2+f_1^2)}}[(g_1(u)f_{11}(u) - f_1(u)g_{11}(u)du^2 + g(u)f_1(u)dv^2)]$,

where II denotes second fundamental form.

Example:

Find the normal curvature of the right helicoid r(u,v)=(ucosv, usinv, cv) at a point on it.

Now r_1=(cosv, sinv,0)

r_2=(-u sinv, u cosv,c)

E=$r_1.r_1$=1, F=$r_1.r_2$=0, G=$r_2.r_2 = u^2 + c^2$

r_{11}=(0,0,0), r_{12}=(-sinv, cosv,0), r_{11}=(-ucosv, -usinv,0)

H=$\sqrt{EG - F^2} = \sqrt{u^2 + c^2}$

N=$\frac{r_1 \times r_2}{H} = \frac{1}{H}(csinv, -ccosv, u)$

L=$N.r_{11} = 0$, M=$N.r_{12} = -\frac{c}{H}$and N=$N.r_{22} = 0$

Thus we have

$$k_n = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2} = \frac{-2cdudv}{H[du^2+(u^2+c^2)dv^2]}, \text{ where } H = \sqrt{u^2 + c^2}$$

Example:

Prove that if L,M,N vanish at all points of a surface, then the surface is the plane.

Let us suppose the surface be a plane surface. Then the surface normal N is a constant vector so that $N_1 = N_2 = 0$ .....(1)

Now L=$-N_1.r_1$, M==$-N_2.r_1$=$-N_1.r_2$, N=$-N_2.r_2$ ...(2)

Using (1) in (2), we have L=M=N=0

Conversely, let us assume that L=M=N=0 and show that the surface is a plane. For this it is enough if we show that the surface normal N is a constant vector.

From hypothesis, we have $N_2.r_1 = N_1.r_2$=0 ...(3)

We know that $N.r_1 = N.r_2$=0 ...(4)

Eqn (3) and (4) together imply N_1 and N_2 are parallel to N. So let them $N_1 = \lambda N$ and $N_1 = \mu N$ where $\lambda and \mu$ are constants. since N.N=1, we have from the above,

N.$N_1 = \lambda$ and N.$N_2 = \mu$ ....(5)

Further from N.N=1, we get N.N_1 =0 and N.N_2=0 ...(6)

Using (6) in (5), we find λ=0, μ=0.

This proves that N_1 =N_2 =0. Thus N is a constant vector which proves the result.

Theorem: ***Meusnier theorem***

If ϕ is the angle between the principal normal n to a curve on a surface and the surface normal N then $k_n = kcos\phi$

Proof:

We know that at any point on a curve on a surface $r'' = k_n + \lambda r_1 + \mu r_2$ ....(1)

Taking dot product with N on both sides of (1), we obtain

$k_n = r''.N$ ....(2)

But we know that $r'' = \frac{dt}{ds} = kn$ ....(3)

Using (3) in (2) we get,

$$k_n kn.N = kcos\phi$$

Thus the normal curvature k_n is the projection on the surface normal N of a vector of length k along the principal normal to the curve.

Example:

Show that the curvature k at a point of the curve of intersection of two surface is given by $k^2sin^2\alpha = k_1^2 + k_2^2 - 2k_1k_2cos\alpha$ where k_1 and k_2 are the normal curvature of the surface in the direction of the curve at P and α is the angle between their normals at the point.

Let P be a point on the curve of intersection C of two surfaces S_1 and S_2 with unit surface normal N_1 and N_2 at P respectively.

By hypothesis $N_1.N_2 = cos\alpha$

Since N_1, N_2 and n are perpendicular to the same tangent vector at P, they are coplanar and lie in the normal plane. If θ is the angle between N_1 and n, then $(\alpha - \theta) or (\alpha + \theta)$ is the angle between N_2 and n. If $k_1 and k_2$ are normal curvature at P on $S_1 and S_2$, we have by meusnier's theorem

$k_1 = kcos\theta$ and $k_2 = kcos(\alpha - \theta)$ ...(1)

Since $k_1 = kcos\theta, sin\theta = \frac{\sqrt{k^2 - k_1^2}}{k}$ ...(2)

We shall eliminate θ between equation (1) and (2)

$k_2 = k[cos\alpha cos\theta + sin\alpha sin\theta]$ ....(3)

Using (2) in (3), we get

$k_2 = k_1cos\alpha + \sqrt{k^2 - k_1^2}sin\alpha$ which gives $(k_2 - k_1cos\alpha)^2 = (k^2 - k_1^2)sin^2\alpha$ which gives on simplification.

$k^2sin^2\alpha = k_1^2 + k_2^2 - 2k_1k_2cos\alpha$ which proves the result.

The result is true if $(\alpha + \theta)$ is the angle between N_2 and n.

Classification of points on a surface:

Depending upon the sign of the normal curvature, we shall classify the points on the surface. To this end let us consider

$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$ ...(1)

Now $LN - M^2 = (N_1.r_1)(N_2.r_2) - (N_2.r_1)(N_1.r_2)$

$=(N_1 \times N_2)(r_1 \times r_2)$ ...(2)

which is not necessarily positive at a point P on a surface. Thus $LN - M^2$ is not always of the same sign, whereas $EG - F^2$ is always positive. Therefore we note that the denominator of k_n is always positive, whereas numerator of k_n is not necessarily positive. Sp the sign k_n depends upon the second fundamental form in the numerator of (1).

The second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$ ...(3) is a quadratic form in du and dv. The discriminant of the quadratic form is $LN - M^2$.

Using the discriminant, we can rewrite the second fundamental form as

$$Ldu^2 + 2Mdudv + Ndv^2 = \frac{1}{L}[(Ldu + Mdv)^2 + (LN - M^2)dv^2]$$

The above quadratic form is positive definite, a perfect square or indefinite according as LN-$M^2 > 0$, LN-$M^2 = 0$ or LN-$M^2 < 0$. So we shall consider the following three particular case.

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Case:1

Let LN-$M^2 > 0$. since the discriminant is positive, the quadratic form is positive at any point P on the surface. Hence k_n has the same sign for all direction at P. In this case the point P is called an elliptic point.

Case:2

Let LN-$M^2 = 0$. since L≠0, the quadratic form becomes(Ldu+Mdv) ^{2}. Hence k_n has the same sign for all direction through P except when $k_n = 0$ which implies $\frac{du}{dv} = -\frac{M}{L}$. In this case the point P is called a parabolic point. The critical directions are called asymptotic directions.

Case:3

Let LN-$M^2 < 0$. The quadratic form is indefinite and k_n does not retain the same sign for all direction at P. It may happen that k_n is zero, positive or negative. If k_n=0, the two direction corresponding to k_n=0 form an angle with respect to the two roots (u_1, v_1) and (u_2, v_2) of the quadratic form. k_n is positive for certain direction lying inside the angular region and negative for directions outside this angular region. The point P is called a hyperbolic point. The two critical directions bounding the angle are called asymptotic directions.

Example:

Show that the point of the paraboloid r=(ucosv, usinv, u^2) are elliptic but the points of the helicoid r=(ucosv, usinv, av) are hyperbolic.

We find L,M,N

For the paraboloid, r=(ucosv, usinv, u^2)

Then $r_1 = (cosv, sinv, 2u)$

$$r_2 = (-ucosv, -usinv, 0)$$

$$r_{11} = (0,0,2), r_{22} = (-ucosv, -usinv, 0), r_{12} = (sinv, cosv, 0)$$

Now HL=$[r_{11}, r_1, r_2]$= $\begin{vmatrix} 0 & 0 & 2 \\ cosv & sinv & 2u \\ -usinv & ucosv & 0 \end{vmatrix}$

=2u

In a similar manner HN=$[r_{22}, r_1, r_2] = 2u^3$and HM=$[r_{12}, r_1, r_2]$=0

Hence LN-M $^2 = \frac{4u^4}{H^2}$ which is always positive so that every point on the paraboloid is elliptic.

For the helicoid r=(ucosv, usinv, av)

$$r_1 = (cosv, sinv, 0)$$

$$r_2 = (-usinv, -ucosv, a)$$

$$r_{11} = (0,0,0), r_{22} = (-ucosv, -usinv, 0), r_{12} = (-sinv, cosv, 0)$$

Using the same formula as in above,

HL=$[r_{11}, r_1, r_2]$=0, HN=$[r_{22}, r_1, r_2] = 0$ and HM=$[r_{12}, r_1, r_2]$=-a

Hence LN-M $^2 = \frac{-a^2}{H^2}$ which is negative so that every point of the helicoid is hyperbolic.

Example:

Show that the anchor ring contains all three types of points namely elliptic, parabolic and hyperbolic on the surface.

Any point on the surface of the anchor ring is

r=[b+acosu)cos v, (b+acosu)sinv, asinu]

where $0 < u \leq 2\pi$ and $0 \leq v < 2\pi$

Using the sign LN-M ^{2}, we shall find the nature of points on the anchor ring.

NOTES

r_1=(-asinu cosv, -asinu sinv, acosu)
r_2=(-(b+acosu)sinv,(b+acosu)cosv, 0)
$r_1 \times r_2$=(b+acosu)[-acosu cosv, -asinv cosv, -asinu]
E=$r_1^2 = a^2$, F=$r_1.r_2$=0, G=$r_2^2 = (b + acosu)^2$
and H=a(b+acosu)
r_{11}=(-acosu cosv, -acosu sinv, -asinu)
r_{12}=(-asinu sinv, -asinu cosv, 0)
r_2=[-(b+acosu) cosv, -(b+acosu) sinv, 0)
L=$\frac{r_{11}.N}{H} = \frac{(b+acosu)^2 acosu}{(b+acosu)a}$
=(b+acosu)cosu

Using the above values of L,M,N, we get
LN-M ^{2}=a(b+acos u)cosu

Since $b > a$, (b+acosu) is positive for all values of u in its domain. So, the sign of LN-M ^{2} is determined only by cosu alone. This leads to the following three cases of u.

Case:1

Let $0 < u < \frac{\pi}{2}$ or $\frac{3\pi}{2} < u < 2\pi$. Then cosu is positive and so LN-M ^{2} is positive. So all those points corresponding to these values of u are elliptic. The points on the torus corresponding to these values of u are at a distance greater than b from the axis of rotation. Hence the points on the torus whose distance from the axis of rotation are greater than b are elliptic points. These are outside points which can be obtained by rotating BCA.

Case:2

Let u=$\frac{\pi}{2}$ or $\frac{3\pi}{2}$. Then cosu=0 which implies LN-M ^{2}=0. Hence all points for which u=$\frac{\pi}{2}$ or $\frac{3\pi}{2}$ are parabolic. As v varies, the point lie on circles of radii b at the top and bottom of the surface obtained by rotating B and A.

Case:3

Let $\frac{\pi}{2} < u < \frac{3\pi}{2}$. Since cos u is negative in this range, LN-M ^{2} is negative. So all points in the redgion are hyperbolic. The points on the torus corresponding to these values of u are at a distance less than b from the axis of rotation. Hence the points on the torus whose distance are less than b are hyperbolic points and these are inside points which are obtained by rotating BDA.

Theorem 2.29

If D_p is the perpendicular distance of a point Q on the surface near the given point P on the surface to the tangent plane at P, then

$$D_p = \frac{1}{2}[Ldu^2 + 2Mdudv + Ndv^2]$$

The second fundamental form at any point P(u,v) on the surface is equal to twice the length of the perpendicular from the neighbouring point Q on the tangent plane at P.

Proof:

Let r(u,v) be the given point P on the surface and let its neighbouring point Q be r(u+du,v+dv) on the surface.

Let d be the perpendicular distance of Q from the tangent plane at P.

Then d=QM=Projection of $\vec{P}Q on$QM
=$\vec{P}Q.N$=[r(u+du,v+dv)-r(u,v)].N

Using Taylor's theorem and omitting higher order infinitesimals,

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$d=[(r_1du+r_2dv)+\frac{1}{2}(r_{11}du^2+2r_{12}dudv+r_{22}dv^2)].N$

Since $N.r_1=0, N.r_2=0$, we have

$d=\frac{1}{2}[r_{11}.Ndu^2+2r_{12}.Ndudv+r_{22}.Ndv^2]$

$d=\frac{1}{2}[Ldu^2+2Mdudv+Ndv^2]$

which proves the geometrical interpretation of the second fundamental form.

Principal curvature:

As the normal curvature k_n at P is a function of $l=\frac{du}{ds}$ and $m=\frac{dv}{ds}$ at P, the normal curvature at P varies as (l,m) changes at P. Hence we seek to find the direction at P along which the normal curvature at P has maximum or minimum values. This leads to the notion of principal curvatures and principal directions at P. Hereafterwards, let us denote k_n by k.

Definition:

If κ_a and κ_b are the principal curvature at a point P on the surface, then the mean curvature denoted by μ is defined as $\mu=\frac{1}{2}(\kappa_a+\kappa_b)$

From the sum of the roots of the equation, we have

$$\mu=\frac{1}{2}(\kappa_a+\kappa_b)=\frac{EN+GL-2FM}{2(EG-F^2)}$$

Definition:

If κ_a and κ_b are the principal curvature, the Gaussian curvature denoted by K is defined as $K=\kappa_a\kappa_b$

From the product of the roots of the equation, we have $K=\kappa_a\kappa_b\frac{LN-M^2}{EG-F^2}$

Note:1

We shall prove later the definition of Gaussian curvature defined by exC and the above definition are equivalent.

Note:2

The value of the Gaussian curvature is independent of the parametric system chosen.

Let us consider the parametric transformation $u=\phi(u',v')$ and $v=\psi(u',v')$

We have

$L'N'-M'^2=J^2(LN-M^2), E'G'-F'^2=J^2(EG-F^2)$

where J is the jacobian of the transformation.

Hence $K'=\frac{L\prime N\prime-M\prime^2}{E\prime G\prime-F\prime^2}=\frac{J^2(LN-M^2)}{J^2(EG-F^2)}=\frac{LN-M^2)}{EG-F^2)}, J\neq 0$

Thus we have $K'=K$, showing that the Gaussian curvature is independent of the parametric system chosen.

Note:3

Using the definition of Gaussian curvature, we can characterise different points on a surface as follows.

From the definition of Gaussian curvature, we have $K=\kappa_a\kappa_b\frac{LN-M^2}{EG-F^2}$

We know that $H^2=EG-F^2$ is always positive.

If K is positive at a point P on a surface, than $LN-M^2>0$ which means that P is an elliptic point. Hence a point on a surface is an elliptic point if and only if two principal curvature at a point P are of the same sign.

If K is negative at a point P on a surface, than $LN-M^2<0$ which means that P is an hyperbolic point. Hence a point on a surface is an hyperbolic point if and only if two principal curvature at a point P are of the opposite sign.

NOTES

If K=0 at a point P on a surface, than $LN - M^2 = 0$ which means that P is an parabolic point. Hence a point on a surface is an parabolic point if and only if atleast one of the principal curvature is zero.

Example:

Find the principal direction and principal curvature at a point on the surface x=a(u+v), y=b(u-v), z=uv.

The position vector of any point P on the surface is

r=[a(u+v),b(u-v), uv]

Then $r_1 = (a, b, v), r_2 = (a, -b, u)$

$r_1 \times r_2$=[b(u+v),a(v-u),-2ab]

$r_{11} = (0,0,0), r_{12} = (0,0,1) = r_{21}, r_{22} = (0,0,0)$

E=$a^2 + b^2 + v^2$, F=$a^2 - b^2 + uv$, G=$a^2 + b^2 + u^2$

L=$N.r_{11}$=0, M=$\frac{r_{12}.N}{H} = \frac{-2ab}{H}$, N=0

Now $LN - M^2 = \frac{-4a^2b^2}{H^2}$ which is always negative.

Hence every point on the surface is a hyperbolic point.

The principal direction are given by

$\begin{vmatrix} dv^2 & -2dudv & du^2 \\ E & F & G \\ 0 & M & 0 \end{vmatrix}$ which gives $Edu^2 - Gdv^2 = 0$, since M≠0

Substituting the value of E and G in above equation,

we get $(a^2 + b^2 + v^2)du^2 - (a^2 + b^2 + u^2)dv^2 = 0$

or $\frac{du}{\sqrt{a^2+b^2+u^2}} = \pm \frac{dv}{\sqrt{a^2+b^2+v^2}}$ giving the principal directions.

The principal curvature are given by the equation

$$(EG - F^2)\kappa^2 - (GL + EN - 2FM)\kappa + (LN - M^2) = 0$$

Substituting the value of E,F,G and L,M,N and simplifying

$$H^4\kappa^2 - 4abH(a^2 - b^2 + uv)\kappa - 4a^2b = 0$$

The principal curvature κ_a and κ_b are the roots of the above equation.

Hence the mean curvature $\mu = \frac{1}{2}\frac{4ab(a^2-b^2+uv)}{H^3}$

The Gaussian curvature K=$\frac{-4a^2b^2}{H^4}$

Example:

Show that all points on a sphere are umbilics.

The representation of a point on a sphere with colatitude u and longitude v as parameter is r=(asinu cosv, asinu sinv, acosu)

For this parametric representation of a point on a sphere we shall find first and second fundamental forms at a point and shoe that $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ so that every point on a sphere is an umbilic.

r_1=(acosu cosv, acosu sinv, -asinu)

r_2=(-asinu sinv, asinu cosv, 0)

E=$r_1.r_1 = a^2$, F=$r_1.r_2$=0, G=$r_2.r_2 = a^2sin^2u$

$r_1 \times r_2 = i(a^2sin^2ucosv) + j(a^2sin^2usinv) + ksinucosu$

and $H^2 = a^4sin^2u$

We shall use the scalar triple product formula to find L,M,N

LH=$[r_{11}, r_1, r_2] = H = \begin{vmatrix} -asinucosv & -asinusinv & -acosu \\ acosucosv & acosusinv & -asinu \\ -asinusinv & asinucosv & 0 \end{vmatrix}$

Hence $L = \frac{-a^3sinu}{a^2sinu} = -a$

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In a similar manner, M=$\frac{[r_{12},r_1,r_2]}{H}$=0

and N=$\frac{[r_{22},r_1,r_2]}{H} = \frac{-a^3 sin^2 u}{a^2 sinu} = -asin^2 u$

Now, $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ gives $\frac{-a}{a^2} = \frac{0}{0} = \frac{-asin^2 u}{a^2 sin^2 u} = -\frac{1}{a}$.

Thus all points on a sphere are umbilics and the principal directions are indeterminate at any point on the sphere.

Lines of curvature:

At each point on the surface, we have two mutually perpendicular directions along which the normal curvature has extreme values. So the question aries whether there exists curves on the surface such that the tangent at each point on the curve on the surface coincide with one of the principal directions. This leads to the notion of lines of curvature.

Definition:

A curve on a given surface whose tangent at each point is along a principal direction is called a line of curvature.

From the above definition, first we note the following properties of the lines of curvature.

i) Differential equation of lines of curvature. Taking l=$\frac{du}{ds}$ and m=$\frac{dv}{ds}$ in equations, the principal directions are given by

(Ldu+Mdv)-κ(Edu+Fdv) =0 ...(1)

(Mdu+Ndv)-κ(Fdu+Gdv) =0 ....(2)

when κ is one of the principal curvatures, we get the equations of the lines of curvature as

$$(L-\kappa E)du + (M-\kappa F)dv = 0$$
$$(M-\kappa F)du + (N-\kappa G)dv = 0$$

ii) Eliminating κ between (1) and (2), the combined equation of the two families of lines of curvature is

$(EM-FL)du^2 + (EN-GL)dudv + (FN-GM)dv^2 = 0$ ...(3)

Since (3) also gives the orthogonal principal directions, the two lines of curvature at any point on the surface are orthogonal. Thus the lines of curvature at any point on the surface form two sets of curves intersecting at right angles.

From the existence of solution of the ordinary differential equation (3), we conclude that these curves cover the surface without gaps in the neighbourhood of every point except umbilics. Since the curves cover the surface simply without gaps, we can take them to be orthogonal parametric curves in our future discussion.

Theorem: *Rodrique's formula*

A necessary and sufficient condition for a curve on a surface to be a line of curvature is $\kappa dr + dN$=0 at each point of the lines of curvature, where κ is the normal curvature in the direction dr of the line of curvature.

Proof:

To prove the necessity of the condition, let the given curve on the surface be a line of curvature. Then the direction (du,dv) at any point P on the curve is a principal direction at (u,v) on the surface given by

$(L-\kappa E)du + (M-\kappa F)dv = 0$ ..(1)

$(M-\kappa F)du + (N-\kappa G)dv = 0$ ...(2)

when κ is one of the principal curvatures at (u,v) Let us write (1) and (2) by grouping the fundamental coefficient of the same kind

(Ldu+Mdv)-κ(Edu+Fdv) =0

(Mdu+Ndv)-κ(Fdu+Gdv) =0 ....(3)

Using the values

L=$-N_1.r_1, M = -N_2.r_1, E = r_1.r_1, F = r_1.r_2$ in the first equation of (3) we obtain

$(N_1du + N_2dv).r_1 + \kappa(r_1du + r_2dv).r_1 = 0$ ....(4)

Now dr=$\frac{\partial r}{\partial u}du + \frac{\partial r}{\partial v}dv = r_1du + r_2dv$ ....(5)

dN=$\frac{\partial N}{\partial u}du + \frac{\partial N}{\partial v}dv = N_1du + N_2dv$ ....(6)

Using (5) and (6) in (4), we get$(dN + \kappa dr)r_1 = 0$ ....(7)

Similarly using the values

M=$-N_1.r_2, N = -N_2.r_2, F = r_1.r_2, G = r_2.r_2$ in the second equation of (3) and simplifying using (5) and (6)

We obtain $(dN + \kappa dr)r_2 = 0$ ....(8)

We claim that $(dN + \kappa dr) = 0$

Since N is a vector of constant modulus1, N^2=1 so that N.dN=0 so that dN is perpendicular to N. That is dN is tangential to the surface. Since dN and dr are tangential to the surface at P, $(dN + \kappa dr)$ is also tangential to the surface. Hence it lies in the plane of vectors r_1 and r_2.

From (7) and (8), we conclude that $(dN + \kappa dr)$ is perpendicular to r_1 and r_2. Therefore $(dN + \kappa dr)$ is parallel to $r_1 \times r_2$ which is the direction of the surface normal. Hence $(dN + \kappa dr)$ is parallel to the surface normal contradicting the fact that $(dN + \kappa dr)$ is tangential to the surface. This contradiction proves that $(dN + \kappa dr)$=0.

To prove the sufficiency of the condition, let us assume that there is a curve on the surface for which $(dN + \kappa dr)$=0 for some function κ at a point P on the surface. On this assumption, we must show that this curve is a line of curvature on the surface having κ for its normal curvature at P. To prove this, it is enough if we show that at each point of the curve, its direction coincides with the principal direction.

Since $(dN + \kappa dr)$=0, we have $(dN + \kappa dr)r_1 = 0$, $(dN + \kappa dr)r_2 = 0$ ....(9)

Hence if (du,dv) is the direction of the curve at a point (u,v), then by retracing the steps, (9) gives the equation

$$(L - \kappa E)du + (M - \kappa F)dv = 0$$

$$(M - \kappa F)du + (N - \kappa G)dv = 0$$

so that the direction at (u,v) coincides with the principal direction.

Further since $dN + \kappa dr$=0, we have $-dN = \kappa dr$

Substituting for dr and dN from (5) and (6), we get

$\kappa(r_1du + r_2dv) = -(N_1du + N_2dv)$ ....(10)

Taking dot product with $r_1du + r_2dv$ on both sides of (10), we have

$\kappa(r_1du + r_2dv).(r_1du + r_2dv) = -(N_1du + N_2dv).(r_1du + r_2dv)$ ....(11)

Using the fundamental coefficients E,F, G and L,M,N

(11) gives $\kappa[Edu^2 + 2Fdudv + Gdv^2] = [Ldu^2 + 2Mdudv + Ndv^2]$

So that $\kappa = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$, which proves that κ is the normal curvature of the curve at P in the direction (du,dv). Since (du,dv) gives the principal direction at P, κ is the principal curvature at P. Therefore the direction at each point of the curve is the principal direction having κ for its normal curvature in the principal direction. Hence the curve must be a line of

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curvature on the surface and this complete the proof of Rodrique's formula.

Theorem

A necessary and sufficient condition that the line of curvature be the parametric curves is that F=0, M=0.

Proof:

Let the lines of curvature be taken as parametric curves. Since they are orthogonal ,F=0. .....(1)

Since F=0, $H^2 = EG > 0$ implies E≠0 and G≠0 ....(2)

The differential equation of the line of curvature is

$(EM - FL)du^2 + (EN - GL)dudv + (FN - GM)dv^2 = 0$ ....(3)

The differential equation of the parametric curve is

dudv=0 ...(4)

Since (3) and (4) are identical, we must have

EM-FL=0 and FN-GM=0 .....(5)

Since F=0, $E > 0$ and G$>$ 0 we get M=0 from (5)

Thus F=0, M=0, the differential equation of the lines of curvature to be parametric curves.

Conversely, if F=0, M=0, the differential equation of the line of curvature (3) reduce to (EN-GL)dudv=0

Since EN-GL≠0, dudv=0 so that the lines of curvature become parametric curves.

Theorem: *Euler theorem*

If κ is the normal curvature in a direction making an angle ψ with the principal direction v=constant, then $\kappa = \kappa_a cos^2\psi + \kappa_b sin^2\psi$ where κ_a and κ_b are principal curvature at the point P on the surface.

Proof:

Since $El^2 + 2Flm + Gm^2$=1, the normal curvature at any point P in the direction (l,m) is

$\kappa = Ll^2 + 2Mlm + Nm^2$ ....(1)

Let us choose the lines of curvature at P as parametric curves.

Then by the theorem, M=0,F=0 so that (1) becomes

$\kappa = Ll^2 + Nm^2$,......(2)

The direction coefficient of the parametric curves v=constant and u=constant are $(\frac{1}{\sqrt{E}}, 0)$ and $(0, \frac{1}{\sqrt{G}})$.

If κ_a and κ_b are the normal curvature along these principal directions, then from (2), we obtain

$\kappa_a = L(\frac{1}{\sqrt{E}})^2 = \frac{L}{E}, \kappa_b = N(\frac{1}{\sqrt{G}})^2 = \frac{N}{G}$

If ψ is the angle between the given direction (l,m) and the principal direction $(\frac{1}{\sqrt{E}}, 0)$, then using the cosine formula cos $\psi = Ell' + F(lm' + l'm) + Gmm'$

We obtain $cos\psi = El\frac{1}{\sqrt{E}} = l\sqrt{E}$ so that l=$\frac{cos\psi}{\sqrt{E}}$ ....(4)

Since the parametric curves are orthogonal, the angle between the direction (l,m) and $(0.\frac{1}{\sqrt{G}})$ is $(90 - \psi)$

Hence $\cos(90 - \psi) = sin\psi = G.m\frac{1}{\sqrt{G}} = m\sqrt{G}$ so that m=$\frac{sin\psi}{\sqrt{G}}$ ....(5)

Using the values of (l,m) in (4) and (5) in (2), we get

$$\kappa = \frac{L}{E}cos^2\psi + \frac{N}{G}sin^2\psi$$

Since $\kappa_a = \frac{L}{E}$, $\kappa_b = \frac{N}{G}$

We obtain $\kappa = \kappa_a cos^2\psi + \kappa_b sin^2\psi$, which completes the proof of Euler's theorem.

Corollary: *Dupin's theorem*

The sum of the normal curvature at any point on the surface in two directions at right angles is constant and equal to the sum of the principal curvature at that point.

Proof:

Let κ_a and κ_b be the principal curvature at any point P on the surface. Let κ_1 and κ_2 be the normal curvature along the directions making angles ψ_1, ψ_2 with the principal direction such that $\psi_2 = \frac{\pi}{2} + \psi_1$

Hence by Euler's theorem, we have

$$\kappa_1 = \kappa_a cos^2\psi_1 + \kappa_b sin^2\psi_1 \text{ ....(1)}$$

$$\kappa_2 = \kappa_a cos^2\psi_2 + \kappa_b sin^2\psi_2 \text{ ....(2)}$$

Since $\psi_2 = \frac{\pi}{2} + \psi_1$, we get from (2)

$$\kappa_2 = \kappa_a cos^2\psi_1 + \kappa_b sin^2\psi_1 \text{ ....(3)}$$

Adding (1) and (3), we get

$$\kappa_1 + \kappa_2 = \kappa_a(cos^2\psi_1 + sin^2\psi_1) + \kappa_b(sin^2\psi_1 + cos^2\psi_1)$$

Hence $\kappa_1 + \kappa_2 = \kappa_a + \kappa_b$

Example:

Show that the meridians and parallels of a surface of revolution are its lines of curvature.

The position vector of any point on the surface of revolution is

r=(u cosv, u sinv, f(u)) .....(1)

We know that the meridian v=constant and the parallel u=constant are parametric curves. By theorem, the parametric curves are lines of curvature if an only if F=0 and M=0. So it is enough if we prove F=0, M=0.

From (1), we have $r_1 = (cosv, sinv, f_1)$ and

r_2=(-usinv, ucosv,0)

Hence $r_1 . r_2$=F=0.

Further $r_1 \times r_2 = (-f_1 ucosv, -f_1 usinv, u)$

and r_{12}=(-sinv, cosv,0), since f_1 is a function of u only.

Now M=$\frac{r_{12}.(r_1 \times r_2)}{H}$

$=\frac{1}{H}[f_1 usinvcosv, -f_1 usinvcosv]$

Since H≠0, M=0

12.4 Check your progress

- State second fundamental form
- Define principal curvature
- State Rodrigue's formula
- State Euler's formula

12.5 Summary

If k_n is the normal curvature of a curve at a point on a surface, then

$k_n = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$ where L=N.r $_{11}$, M=N.r $_{12}$, N=N.r $_{22}$ and E,F,G are the first fundamental coefficients.

If ϕ is the angle between the principal normal n to a curve on a surface and the surface normal N then $k_n = kcos\phi$

NOTES

The second fundamental form at any point P(u,v) on the surface is equal to twice the length of the perpendicular from the neighbouring point Q on the tangent plane at P.
all points on a sphere are umbilics.

12.6 Keywords

Second fundamental form: The quadratic form $Ldu^2 + 2Mdudv + Ndv^2$ is called the second fundamental form of the surface and L,M,N which are functions of u,v are called second fundamental coefficients.

Principal curvature: As the normal curvature k_n at P is a function of l=$\frac{du}{ds}$ and m=$\frac{dv}{ds}$ at P, the normal curvature at P varies as (l,m) changes at P. Hence we seek to find the direction at P along which the normal curvature at P has maximum or minimum values. This leads to the notion of principal curvatures and principal directions at P. Hereafterwards, let us denote k_n by k.

Mean Curvature: If κ_a and κ_b are the principal curvature at a point P on the surface, then the mean curvature denoted by μ is defined as $\mu = \frac{1}{2}(\kappa_a + \kappa_b)$. From the sum of the roots of the equation, we have

$$\mu = \frac{1}{2}(\kappa_a + \kappa_b) = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

Lines of curvature: At each point on the surface, we have two mutually perpendicular directions along which the normal curvature has extreme values. So the question aries whether there exists curves on the surface such that the tangent at each point on the curve on the surface coincide with one of the principal directions. This leads to the notion of lines of curvature.

12.7 Self Assessment Questions and Exercises

1. For the surface r=(u cosv, u sinv, f(u)), find
   i) The principal direction and principal curvature at any point on the surface.
   ii) The normal curvature along a direction making an angle $\frac{\pi}{4}$ with the merdian of the surface.
2. Find the position vector of any point on a surface generated by tangents to a twisted curve. Obtain the principal direction and principal curvature at any point on the suface.
3. Find the Gaussian curvature of the conoid r(u,v)= (u cosv, u sinv, cos 2v).
4. Find the Gaussian curvature and mean curvature of the helicoid r(u,v)=(u cosv, u sinv, f(u)+cv). Show that the Gaussian curvature is a constant along a helx. Find also the mean curvature along the helix.
5. Show that the Gaussian curvatur and mean curvature on the surface x=u+v, y=u-v, z=uv at the point x=2, y=0, z=1 are K=-$\frac{1}{16}$ and μ=$\frac{1}{8\sqrt{2}}$.
6. Find the nature of the points on the following surface.
   i) r(u,v)=(u,v,u^2-v^2) ii) r(u,v)=(u,v,u^2+v^3)
7. Show that all points on a tangent surface of a curve C are parabolic.
8. Show that the points of the paraboloid r(u,v) = (u cosv, v sinv, u^2) are elliptic but the points of the helicoid r(u,v) =(u cosv, u sinv, v) are hyperbolic.
9. Find the differential equation of the lines of curvature of the surface

generated by i) binormals and ii) principal normals of a twisted curve.

10. Find the line of curvature on a plane.
11. Show that any curve on a sphere is a line of curvature.
12. If a plane or a sphere cuts a surface everywhere at a constant angle, prove that a curve of intersection is a line of curvature on the surface.

12.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010).

UNIT XIII DEVELOPABLES

NOTES

Structure

13.1 Introduction

This chapter deals with the concept of developable surfaces. In the plane,we find the envelope of straight lines. The extension of the notion of envelope of straight lines to envelope of planes in space leads to what are called developable surfaces. When we specialise to different planes such as osculating plane at a point on a space curve, we different developable surfaces.

13.2 Objectives

After going through this unit, you will be able to:

- Define developable surface.
- Define characteristic line
- Derive the properties of developable surface
- Define osculating plane of surface

13.3 Developables

Developable surfaces:

In a plane, we find envelope of straight lines. For example the envelope of the normals to a curve leads to the evolute in the plane. The extension of the notion of envelope of straight lines to envelope of planes in space leads to what are called developable surfaces. When we specialise to different planes such as osculating plane at a point on a space curve, we get different developable surface.

Definition:

The envelope of one parameter family of planes is called a developable surface or a developable.

Definition:

The line of intersection of the two consecutive planes is called the characteristic line.

Definition:

When the planes f(u)=0, f(v) and f(w)=0 intersect at a point, the limiting position of the point of intersecting of the three planes as v→u and $w \to u$ is called the characteristic point corresponding to the plane u.

When v→u and $w \to u$, it gives rise to two characteristic lines. So when v,w→u, these two characteristic line will pass through the characteristic point so that the characteristic point can be defined as the ultimate point of intersection at the two consecutive characteristic lines.

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Theorem

The characteristic point of the plane u is determined by the equations *r.a=p,* $r.\dot{a} = \dot{p}$ *and r.*$\ddot{a} = \ddot{p}$ *...(1)*

Proof:

Let u,v,w be three neighbouring points such that f(u)=0, f(v)=0, f(w)=0.

Hence by Rolle's theorem there exist points u_1, u_2 such that $u < u_1 < v, v < u_2 < w$ foe which $\dot{f}(u_1) = 0$ and $\dot{f}(u_2) = 0$

Using Rolle's theorem again, there is a point u_3 such that $u_1 < u_3 < u_2$ for which $\ddot{f}(u_3)$=0.

Hence when u_1, u_2, u_3 all tends to u, we get

$$f(u)=0,\ \dot{f}(u) = 0 \text{ and } \ddot{f}(u) = 0$$

or equivalently we get

$$r.a=p,\ r.\dot{a} = \dot{p} \text{ and } r.\ddot{a} = \ddot{p}$$

Example:

Let us consider a cylinder with its generators parallel to a. Then it has a constant tangent plane along a generator. Since the tangent plane along the generator depends upon only one parameter a, a cylinder can be considered as a developable surfaces as the envelope of the single parameter tangent planes along the generators. Since a is a constant vector $[a, \dot{a}, \ddot{a}]$=0 so that the equation (1) does not have a solution. Therefore the characteristic point does not exist in this case.

Example:

A cone is a developable surface enveloped by the constant tangent planes along the generators of the cone. The generators of the cone are the characteristic lines. Since all the planes through the generator pass through the vertex of the cone, the vertex is the characteristic point of the surface.

Example:

Let us consider a family of planes forming a pencil. Then since the envelope of the planes is the axis, the developable is the axis of the pencil. Hence the point of intersection of the characteristic lines are indeterminate.

Definition:

The locus of the characteristic points is called the edge of regression of the developable.

Theorem

The tangent to the edge of regression are the characteristic lines of the developable.

Proof:

Let the developable be the envelope of the one parameter family of planes r.a=p.

Let r=r(s) be the position vector of any point P on the edge of regression. Since P is on the edge of regression it is a characteristic point.

Hence by the above theorem, it is given by the equation

r.a=p....(1)

$r.\dot{a} = \dot{p}$....(2)

r.$\ddot{a} = \ddot{p}$ ....(3)

where the solution r is a function of u.

Diff. (1) w.r to u

$$\frac{dr}{ds}\frac{ds}{du}.a + r.\dot{a} = \dot{p}$$

NOTES

Since $\frac{dr}{ds}$ gives the tangent t to the edge of regression, using (2) in the above equation $(t\dot{s}).a$=0. Since $\dot{s}$ ≠0, we get t.a=0 .....(4)

Diff (2) w r tou

$$(t\dot{s}).\dot{a}+r.\ddot{a}=\ddot{p}$$

Using (3) in the above equation, we have $(t\dot{s}).\dot{a}$=0

Since $\dot{s}$ ≠=0, we have $t.\dot{a}=0$ ....(5)

(4) and (5) show that the tangent t to the edge of regression is parallel to $a\times\dot{a}$.

Since the characteristic line lies in both the planes (1) and (2),it is perpendicular to both a and $\dot{a}$ and hence it is parallel to $a\times\dot{a}$.

Since the tangent to the edge of regression and the characteristic line are parallel to the same vector to $a\times\dot{a}$ and pass through r(u), the tangent to the edge of regression is the characteristic line of the developable.

Theorem

The osculating plane at any point on the edge of regression is the tangent plane to the developable at that point.

Proof:

First note that the edge of regression is a curve on the developable surface.

From the equation (4) of the previous theorem, t.a=0 .....(1)

Diff (1) w.r to u,

$$(\frac{dt}{ds}\frac{ds}{du}).a+t.\dot{a}=0$$

Using (5) of the previous theorem, we have ()$\kappa n\dot{s}.a$=0

Since $\kappa\dot{s}$ ≠0, we get from the above step, n.a=0 ....(2)

From (1) and (2) we see that a is perpendicular to both t and n. Hence a is parallel to t× n=b, the binormal at P on the edge of regression.

Thus the osculating plane at P is identical with the corresponding plane of the parameter family. But each plane of the family is a tangent plane to the developable surface so that the osculating plane at any point on the edge of regression is the tangent plane to the developable surface at P.

Theorem

A developable consists of two sheets which are tangent to the edge regression along the sharp edge.

Proof:

Let O be the point s=0 on the edge of regression curve C and choose the orthogonal triad at O as the rectangular coordinate axes O(x,y,z). If R is the position vector of the point P(x,y,z) on the developable with respect to O, then we have

R=xt+yn+zb .....(1)

The position vector of any point on the developable surface is R(s,v)=r(s)+vt(s) .....(2)

Let us expand r(s) and t(s) of (2) in power series of s by taylor series at the origin O. Then we get

R=$s\frac{dr}{ds}+\frac{s^2}{2}\frac{d^2r}{ds^2}+\frac{s^3}{3!}\frac{d^3r}{ds^3}+O(s^4)+v\{t+s\frac{dt}{ds}+\frac{1}{2}s^2\frac{d^2t}{ds^2}+O(s^3)\}$ ....(3)

Using Frenet-serret formulae we simplify r(s) and t(s) as follows

r(s)=$st+\frac{s^2}{2}t'+\frac{s^3}{6}(\kappa'n+n'\kappa)+O(s^4)$

=$st+\frac{s^2}{2}\kappa n+\frac{s^3}{6}(\kappa'n+\kappa\tau b-\kappa^2t)+O(s^4)$ ....(4)

vt=v$\{t+s\kappa n+\frac{1}{2}s^2(\kappa'n+\kappa\tau b-\kappa^2t)+O(s^3)\}$

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Using (4) and (5) in (3), we obtain, R(s,v)=$st+\frac{s^2}{2}\kappa n+\frac{s^3}{6}(\kappa' n+\kappa\tau b-\kappa^2 t)+O(s^4)+v\{t+s\kappa n+\frac{1}{2}s^2(\kappa' n+\kappa\tau b-\kappa^2 t)+O(s^3)\}$ .....(6)

Now equating the coefficient of t in (1) and (6), we have

x=s-$\frac{1}{6}s^3\kappa^2+O(s^4)+v\{1-\frac{s^2}{2}\kappa^2+O(s^3)\}$ .....(7)

Let us find the point of intersection of the developable with the normal plane x=0 at O. The value of the parameter of the point of intersection of the normal plane with the surface is obtained from x=0 in (7). Hence when x=0, we have from (7)

v=-$(s-\frac{1}{6}s^3\kappa^2+\dots)(1-\frac{s^2}{2}\kappa^2+\dots)^{-1}$

=-$(s-\frac{1}{6}s^3\kappa^2+\dots)(1+\frac{s^2}{2}\kappa^2+\dots)$

=$s-\frac{1}{3}s^3\kappa^2+O(s^4$

Substituting the above values of v in (3), we get the position vector of the point of intersection of the surface and the normal plane at O.

Then we have

R(s,v)=$st+\frac{s^2}{2}\kappa n+\frac{s^3}{6}(\kappa' n+\kappa\tau b-\kappa^2 t)+O(s^4)+(s-\frac{1}{3}s^3\kappa^2+O(s^4)[t+s\kappa n+\frac{1}{2}s^2(\kappa' n+\kappa\tau b-\kappa^2 t)]$ .....(8)

Equating the coefficients of t,n,b in (1) and (8), we get

x=0, y=$\frac{1}{2}s^2\kappa+O(s^3)$, z=$-\frac{1}{3}\kappa\tau s^3+O(s^4)$ ....(9)

which is a curve in the normal plane.

Eliminating s between the equation (9), we get

$z^2=-\frac{8}{9}\frac{\tau^2}{\kappa}y^3$ .....(10),upto first approximation.

Equating (10) shows that the intersection of the developable with the normal plane of the edge of regression has a cusp whose tangent is along y=0 and z=0 which is the principal normal. Thus two sheets of the developable surface are thus tangent to the edge of regression along a sharp edge.

Example:

Find the equation of the developable surface which has helix r=(acosu, asinu, cu) for its edge of regression.

Since the developable surface is generated by the tangent to the edge of regression, let us find the tangent vector to the edge of regression.

Now, $\frac{dr}{du}=\frac{dr}{ds}\frac{ds}{du}=(-asinu,acosu,c)$

Since $\frac{dr}{ds}$=t, we have $(\frac{ds}{du})^2=(a^2+c^2)$ so that $\frac{ds}{du}=\sqrt{a^2+c^2}$

Hence t=$\frac{1}{\sqrt{a^2+c^2}}(-asinu,acosu,c)$

If R is the position vector of a point on the surface, then

R=r+vt=r+$\frac{v}{\sqrt{a^2+c^2}}(-asinu,acosu,c)$

Substituting for r, we get the developable surface as

R(u,v)=a$\{a(cosu-\frac{v}{\sqrt{a^2+c^2}}sinu),a(sinu+\frac{vcosu}{\sqrt{a^2+c^2}}sinu),c(u+\frac{v}{\sqrt{a^2+c^2}})\}$

Developables associated with space curves:

At each point of a space curve, we have three planes viz. osculating plane, normal plane and rectifying plane. All the three planes contain only the arc-length s as parameter so that they are one parameter family of planes. So as in the previous section, we can find the envelopes of these single

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parameter family of planes. These lead to the three kinds of developable surface viz. (i) the osculating developable (ii) the polar developable (iii) the rectifying developable. In each case we find the edge of regression of developable and finally, we obtain a criterian for a surface to be a developable surface.

Definition:

The envelope of the family of osculating planes of a space curve is called an osculating developables. Its characteristic lines are tangents to the curve and hence this developable is also called the tangential developable.

Theorem

The space curve is itself the edge of regression of the osculating developable.

Proof:

Let r=r(s) be the equation of the space curve and R be the position vector of any point on the osculating plane. Then the equation of the osculating plane is f(s)=(R-r).b=0 .....(1)

Since r and b are functions of s, (1) is a single parameter family of planes. To find the equation of the osculating developable, let us differentiate (1) with respect to s, then

$$f'(s) = (R - r).b' - r'.b = 0$$

Using Serret-Frenet formula, we get

$$(R - r).(-\tau n) - t.b = 0$$

Since t.b=0 and $\tau \neq 0$, we get (R-r).n=0 ....(2),

which is the rectifying plane at r=r(s) of the curve.

Since the characteristic line is the intersection of (1) and (2), it is the tangent to C at r=r(s). Thus the characteristic lines to the osculating developable are tangent to C.

To find the edge of regression, let us differentiate (2) with resprct to s,

$(R - r).n' - r'.n = 0$ giving $(R - r).(\tau b - \kappa t) - t.n$=0

Since $\kappa \neq 0$ and t.n=0, using (1) in the last equation, we get

(R-r).t=0 ....(3)

which is the normal plane at r=r(s) to C.

The point of intersection of (1), (2) and (3) is the characteristic point and its locus is the edge of regression. Since (1), (2) and (3) intersect at r=r(s) on the curve, every point r=r(s) of the curve is the characteristic point so that the characteristic points coincide with every point on the curve.

Hence the edge of regression is the given curve.

Definition:

The envelope of the normal planes to the space curve is called the polar developables.

Theorem

The edge of regression of the polar developable of a space curve is the locus of the centre of spherical curvature.

Proof:

Let r=r(s) be the given space curve and R be any point on the normal plane. Then the equation of the normal plane is (R-r).t=0 .....(1)

Since r and t are function of single parameter s, (1) is a single parameter family of planes whose envelope is the polar developable. To find the edge of regression, we shall find the characteristic point from the following equations.

Differentiate (1) with respect to s, we have

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$$(R-r).t'-r'.t=0$$
Since $t'=\kappa n. r'=t$ and t.t=1, we have
$(R-r).\kappa n-1=0$ so that (R-r).n=$\frac{1}{\kappa}=\rho$ ....(2)
Differentiate (2) with respect to s,
(R-r).$n'-r'.n=\rho'$
Since $n'=\tau b-\kappa t and r'$=t, we have
$$(R-r).(\tau b-\kappa t)-t.n=\rho'$$
Using (1) and t.n=0, we obtain
$(R-r).b=\frac{\rho'}{\tau}=\rho'\sigma$ ....(3)
The point of intersection (1), (2) and (3) is the characteristic point and its locus is the edge of regression of the polar developable.
Since (R-r) is orthogonal to t, (R-r) lies in the plane of n and b so that we can take
$R-r=\lambda n+\mu b$ or $R=r+\lambda n+\mu b$ .....(4)
where we determine the scalars $\lambda and \mu$.
Taking dot product with n on both sides of (4), we get
(R-r).n=λ ....(5)
Comparing (2) and (5), we obtain $\lambda=\rho$
Taking dot product with b on both sides of (4), we get
$$(R-r).b=\lambda n.b+\mu b.b$$
Since b.b=1, n.b=0, using eqn (3), we have $\mu=\sigma\rho$
Substituting the values of $\lambda and \mu$ in (4), we get
$R=r+\rho n+\sigma\rho' b$ ....(6)
which gives the position vector of the characteristic point but we know that R gives the position vector of the centre of spherical curvature. Thus the characteristic point of the polar developable coincide with the centre of spherical curvature.
Hence the edge of regression of the polar developable is the locus of the centre of spherical curvature.

Definition:
The envelope of the family of the rectifying planes of a space curve is called rectifying developable.

Theorem
The edge of regression of the rectifying developable has the equation
R=$r+\frac{\kappa(\tau t+\kappa b)}{\kappa'\tau-\kappa\tau'}$

Proof:
Let r=r(s) be the given space curve and R be any point on the rectifying plane. The equation of the rectifying plane is (R-r).n=0 ...(1)
Since r and n are functions of s, (1) is a single parameter family of planes whose envelope is the rectifying developable.
We shall find the edge of regression from the following equation.
Differentiate (1), we obtain
$$(R-r).n'-r'.n=0$$
Since $n'=\tau b-\kappa t, r'=t$ and t.n=0, we get
(R-r).$(\tau b-\kappa t)$=0 ....(2)
Differentiate (2) with respect to s, we get
(R-r).$[\tau b'+\tau' b-\kappa t'-\kappa' t]-r'.(\tau b-\kappa t)=0$
Since $b'=-\tau n, t'=\kappa t$ and $r'=t,$ we get
(R-r).$[-\tau^2 n+\tau' b-\kappa^2 n-\kappa' t]-t.(\tau b-\kappa t)=0$

NOTES

Since t.b=0, t.t=1, using (1), we get from the above equation,

(R-r).$(\tau' b - \kappa' t) + \kappa$=0 ....(3)

The edge of regression is the point intersection of (1), (2) and (3). From (1) and (2), (R-r) is perpendicular to n and $\tau b - \kappa t$.

So (R-r) is parallel to $n \times (\tau b - \kappa t) = \tau t + \kappa b$

Hence we can take (R-r)=$\lambda(\tau t + \kappa b)$ ....(4)

where λ is a scalar to be determined.

Since t.t=1, b.b=1, t.b=1, using (3) in the above equation,

$-\kappa = \lambda(-\tau\kappa' + \kappa\tau')$ so that $\lambda = \frac{\kappa}{\kappa'\tau - \tau'\kappa}$

Using this value of λ in (4)

$(R - r) = \frac{\kappa}{\kappa'\tau - \tau'\kappa}(\tau t + \kappa b)$ which gives the edge of regression.

Theorem

Every space curve is a geodesic on its rectifying developable.

Proof:

The position vector R on the rectifying developable of the given curve r=r(s) is R=r(s)+u$(\tau t + \kappa b)$ ......(1)

where u and s are parameter of the surface.

Let us find the surface normal N to the surface.

From (1), $R_1 = \frac{\partial R}{\partial u} = \tau t + \kappa b$ and $R_2 = t + u[\tau' t + \tau t' + \kappa' b + \kappa b']$

=$t + u[\tau' t + \tau\kappa n + \kappa' b - \kappa\tau n]$

=$t(1 + \tau' u) + (u\kappa' b)$

Hence $R_1 \times R_2 = \tau t + \kappa b \times [t(1 + \tau' u) + (u\kappa' b)]$

=$\kappa(1 + \tau' u) - \tau u\kappa' n$

u=0 on the curve r=r(s) so that $R_1 \times R_2 = \kappa n$ ....(2)

Since $R_1 \times R_2$=HN, we get from (2), HN=κn.

Hence at each point of the curve, the vector N and n are parallel so that (1) is a geodesic on the surface which is the rectifying developable in this case.

Theorem

A necessary and sufficient condition for a surface to be a developable is that is Gaussian curvature shall be zero.

Proof:

To prove the necessity of the condition, let us assume that the surface is a developable and show that its Gaussian curvature is zero.

If the developable is a cylinder or a cone, the Gaussan curvature is zero, since at each point on the cone or cylinder one of the principal directions is the gernerating straight line whose curvature is zero.

Hence excluding these two cases of developables, we are left with developable in general. Since a developable can be considered as the osculating developable of its edge of regression, it is generated by the tangent to the edge of regression.

Let r=r(s) be the equation of the edge of regression on the developable. If P is any point on this curve, then the tangent at P is the characteristic line of the developable.

R can be taken as R(s,v)=r(s)+vt(s) ....(1)

For this surface, let us find E,F,G,L,M and N and show that its Gaussian curvature $K = \frac{LN - M^2}{EG - F^2}$=0

Using the suffixes 1 and 2 for differentiate with respect to s and v respectively.

NOTES

$$R_1 = \frac{\partial R}{\partial s} = \frac{dr}{ds} + v\frac{dt}{ds} = t + v\kappa n$$
$$R_2 = \frac{\partial R}{\partial v} = t(s)$$
$$R_{11} = \frac{dt}{ds} + v\kappa' n + v\kappa n' = \kappa n + \kappa' n + v\kappa(\tau b - \kappa t)$$

$R_{12} = R_{21} = \kappa n$, R_{22}=0

E=$1 + v^2\kappa^2$, F=$(t + v\kappa n)$.t=1,

G=t.t=1, $H^2 = EG - F^2 = v^2\kappa^2$

N=$\frac{R_1 \times R_2}{H} = \frac{-v\kappa b}{v\kappa} = -b$

L=N.$R_{11} = -b.[\kappa n + v\kappa' n + v\kappa(\tau b - \kappa t)] = -v\kappa\tau$

M=N.$R_{12} = -b.(\kappa n)$=0,

N=N.R_{22}=-b.0=0

Hence using the above values, the Gaussian curvature,

$K = \frac{LN - M^2}{EG - F^2}$=0

and this proves the necessity of the condition.

To prove the converse, let us assume that K=0 for a surface r=r(u,v) and show that it is a developable surface.

To this end, we have to show that r=r(u,v) is generated by a single parameter family of planes.

Since L=$-r_1.N_1, M = -r_1.N_2 = -N_1.r_2, N = -r_2.N_2$,

$$LN - M^2 = (r_1.N_1)(r_2.N_2) - (N_1.r_2)(r_1.N_2)$$

=$(r_1 \times r_2).(N_1 \times N_2)$

Since $r_1 \times r_2$=HN, we have $LN - M^2 = HN.(N_1 \times N_2) = H[N, N_1, N_2]$

As $H \neq 0$, $\kappa = 0$ implies $LN - M^2$ =0 so that $[N, N_1, N_2]$=0

Since N.N=1, we have N.N_1=0 and $N.N_2 = 0$

So N is perpendicular to both N_1 and N_2. Hence N is parallel to $N_1 \times N_2$.

Thus $[N, N_1, N_2] = N.(N_1 \times N_2)$ cannot be zero unless N_1=0 0r N_2=0 or N_1 is parallel to N_2. So $[N, N_1, N_2]$=0 under the given condition implies the following two cases.

(i) N_1=0 0r N_2=0 (ii) $N_1 = \mu N_2$

Case:1

Let us consider $N_2 = 0$ only, since a similar argument is true for N_1=0.

The equation of the tangent plane at r=r(u,v) on the surface is

(R-r).N=0 .....(2)

where R is the position vector of any point on the tangent plane and R is independent of u,v.

Differentiate (2) partially with respect to v, we have

$\frac{\partial}{\partial v}[(R - r).N] = -r_2.N + (R - r).N_2$ ....(2)

Now $N_2 = 0$, $r_2.N = 0$.

Hence equation (3) gives $\frac{\partial}{\partial v}[(R - r).N]$=0 so that (R-r).N is independent of v.

Thus the surface is the envelope of a single parameter family of planes and hence it is a developable.

Case:2

In this case, when $N_1 = \mu N_2$, let us consider a suitable change of parameter from u,v to u', v' so that we can use the previous case.

Let the transformation be $u = u' + v'$ and $v = u' - \mu v'$

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This is a proper parametric transformation which preserves surface normal N.

Now $N_1' = \frac{\partial N}{\partial u'} = \frac{\partial N}{\partial u}.\frac{\partial u}{\partial u'} + \frac{\partial N}{\partial v}.\frac{\partial N}{\partial u'} = N_1 + N_2 \neq 0$

$$N_2' = \frac{\partial N}{\partial v'} = \frac{\partial N}{\partial u}.\frac{\partial u}{\partial v'} + \frac{\partial N}{\partial v}.\frac{\partial N}{\partial v'} = N_1 - \mu N_2$$

After the transformation N_1' and N_2' are not parallel as seen from the above equations. Since $N_2' = 0$, as in the previous case, the tangent plane at P is a single parameter family of planes so that the given surface is a developable surface.

Example

Find the osculating developable of the circular helix r=(acosu, asinu, cu)

Since the edge of regression of the osculating developable is the curve itself, the given helix itself is the edge of regression. Hence we have to find the developable surface having (1) as the edge of regression which is the same as example :4 in the previous section.

Example

Find the radii of principal curvature at a point of a tangential developable surface.

Let the equation of the tangential developable be

R(s,v)=r(s)+vt(s) ......(1)

By previous theorem, we can find

E=1 $+ v^2\kappa^2$, G=1, F=1, L=$-v\kappa\tau$, M=0, N=0, and H=vκ ....(2)

where κ, τ belong to the edge of regression.

The equation giving the principal curvature is

$$H^2\kappa_n^2 - (EN - 2FM + GL)\kappa_n + LN - M^2 = 0 \text{ ....(3)}$$

Using (2) in (3), we get

$$v^2\kappa^2\kappa_n^2 + \kappa_n v\kappa\tau = 0 \text{ or } v\kappa\kappa_n[v\kappa\kappa_n + \tau] = 0$$

Hence the principal curvature are $\kappa_a = 0$, $\kappa_b = -\frac{\tau}{v\kappa}$

13.4 Check your progress

- Define developable surface
- Define edge of regression
- State the properties ofdevelopable surface
- Define osculating developable surface

13.5 Summary

- The envelope of one parameter family of planes is called a developable surface or a developable.
- The characteristic point of the plane u is determined by the equations r.a=p, $\mathrm{r}.\dot{\mathrm{a}} = \dot{\mathrm{p}}$ and r.$\ddot{\mathrm{a}} = \ddot{\mathrm{p}}$
- The osculating plane at any point on the edge of regression is the tangent plane to the developable at that point.
- A developable consists of two sheets which are tangent to the edge regression along the sharp edge.

13.6 Keywords

Developable: The envelope of one parameter family of planes is called a developable surface or a developable.

Characteristic line: The line of intersection of the two consecutive planes is called the characteristic line.

Characteristic Point: When the planes f(u)=0, f(v) and f(w)=0 intersect at a point, the limiting position of the point of intersecting of the three planes as v→u and $w \to u$ is called the characteristic point corresponding to the plane u.

Osculating Developable: The envelope of the family of osculating planes of a space curve is called an osculating developables.

Tangential Developable: The characteristic lines of osculating developable are tangents to the curve and hence this developable is also called the tangential developable.

13.7 Self Assessment Questions and Exercises

1. Find the direction conjugate to u=constant on a surface r(u,v)= (u cosv, u sinv,f(u))
2. Obtain the differential equation of the asymptotic lines on the surface of revolution r(u,v)= (u cosv, u sinv, f(u))
3. Find the asymptotic lines on the paraboloid of revolution z=x^2+y^2.
4. Prove that the generators of a ruled surface are asymptotic lines.
5. Find the condition for the asymptotic lines to be orthogonal.
6. Prove that the normal curvature in a direction perpendicular to an asymptotic line is twice the mean normal curvature.

13.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).

2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010).

NOTES

UNIT XIV
DEVELOPABLES ASSOCIATED WITH CURVES ON SURFACES

Structure

13.1 Introduction

In this chapter, the concept of developables associated with curves on surfaces are explained. Also the theorem of Monge formed by surface normals along the curve and Rodrique's formula are also derived. Some properties of developable associated with curves are established and some problems are given.

13.2 Objectives

After going through this unit, you will be able to:

- Derive the Monge's form of surface.
- Derive the Rodrique's formula
- Solve the problems in developables associated with curves on surfaces.

13.3 Developables associated with curves on surfaces.

Theorem: ***Monge's theorem***

A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Proof:

Let r=r(s) be a curve on the surface r=r(u,v). Let N be the unit surface normal at a point r=r(s) on the curve so that N can be considered as a function of s only. We shall prove the theorem in the following two steps.

Step:1

In this step, we prove that the normals to the surface r=r(u,v) along the curve r=r(s) form a developable if and only if $[t, N, N']$=0.

If R is the position vector of any point Q on the surface normal along the curve, then R is a point on the developable generated by the normals and it can be taken as R(s,v)=r(s)+vN(s), where v is the distance PQ along the surface normal at P.

The surface generated by the surface normals is a developable if and only if its Gaussian curvature is zero at every point. This implies that $LN - M^2$=0 at every point of the surface. Hence let us find L,M,N for the surface.

$$R_1 = \frac{\partial R}{\partial s} = \frac{dr}{ds} + v\frac{dN}{ds} = t + vN',$$
$$R_{11} = \kappa n + vN''$$

$$R_2 = \frac{dR}{dv} = N,$$
$$R_{12} = R_{21} = N', R_{22} = 0$$

Now L=$R_{11}.N = [\kappa n + vN''].N \neq 0$

HM=$[R_{12}, R_1, R_2] = [N', t + vN', N]$

So we have HM=$[N', t, N] + v[N', N', N]$ ....(1)

Since N$'$ is equal to $N', [N', N', N] = 0$ .....(2)

As $H \neq 0$, using (2) in (1),

M=$\frac{1}{H}[N', t, N] = \frac{1}{H}[t, N, N']$

Since R_{22}=0 and $H \neq 0$, HN=$[R_{22}, R_1, R_2]$=0 so that N=0

As $L \neq 0$, and N=0, $LN - M^2$=0 if and only if M=0 which gives $[t, N, N']$=0.

Step:2

To prove the theorem, it is enough if we show that $[t, N, N']$=0 is the necessary and sufficient condition for r=r(s) to be a line of curvature on the surface r=r(u,v).

To prove the necessity of the condition, let us assume that r=r(s) is a line of curvature on r=r(u,v). Then we have by Rodrique's formula

$\kappa \frac{dr}{ds} + \frac{dN}{ds} = 0$ which gives $N' = -\kappa t$.....(3)

Using (3), we have $[t, N, N'] = [t, N, -\kappa t]$=0

Hence the surface normal along the curve r=r(s) form a developable surface.

Conversely, assuming $[t, N, N']$=0, we show that r=r(s) is a line of curvature on the surface.

Now $[t, N, N']$=0 implies $[t, N', N]$=0 which is the same as $(t \times N').N = 0$.

Further $N \neq 0$ and $N^2 = 1$ so that $N.N'$=0 showing N' is perpendicular to N. That is N' is in the tangent plane at P.

Hence $t \times N'$ is parallel to N. So if $t \times N' \neq 0$. $(t \times N').N$ cannot be zero because two non zero parallel vectors $t \times N'$ and N will have scalar product different from zeros. So we conclude that $(t \times N').N$=0 implies $t \times N'$=0 which is true if and only if one vector is a scalar multiple of the other.

So we can take $N' = -\kappa t$ for some constant κ.

Now $N' = -\kappa t$ gives $\frac{dN}{ds} = -\kappa \frac{dr}{ds}$ or dN+κdr=0 which is the Rodrique's formula characterising the line of curvature. Thus r=r(s) is a line of curvature on the surface.

This completes the proof.

Theorem

Let C be a curve r=r(s) lying on the surface r=r(u,v) and P be any point on C. Then the characteristic line at P of the tangential developable of C is in the direction conjugate to the tangent to C at P.

Proof:

If N is the surface normal at P, then the equation of the tangent at P is (R-r).N=0 .....(1)

Diff w.r to s and using t.N=0, we get

$(R - r).\frac{dN}{ds}$=0 ....(2)

Since N is a function of (u,v), we have

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$$\frac{dN}{ds} = \frac{\partial N}{\partial u}.\frac{du}{ds} + \frac{\partial N}{\partial v}.\frac{dv}{ds} = N_1u' + N_2v' \text{ ....(3)}$$

where (u', v') gives the direction of the tangent at P.
Using (3) in (2), we get

$$(R - r).(N_1u' + N_2v')\text{=0 ....(4)}$$

The characteristic line is the intersection of the plane (1) and (4). If (l,m) are the direction coefficient of the characteristic line at r=r(s), then (R-r)=$lr_1 + mr_2$ ....(5)
where r_1 and r_2 are the tangential components of (R-r) at P.
Using (5) in (4), we have $(lr_1 + mr_2).(N_1u' + N_2v')$=0
Expanding the above equation

$$(N_1.r_1)lu' + (N_2.r_1)lv' + (N_1.r_2)mu' + (N_2.r_2)mv' = 0$$

Hence $Llu' + M(lv' + mu') + Mmv'$=0 which is precisely the condition for the direction (l,m) to be conjugate to the direction (u', v') which is the direction of the tangent at P.

14.4 Check your progress

- Define Monge's form.
- Derive the necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable

14.5 Summary

- A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.
- Let C be a curve r=r(s) lying on the surface r=r(u,v) and P be any point on C. Then the characteristic line at P of the tangential developable of C is in the direction conjugate to the tangent to C at P.

14.6Keywords

Monge's form: The necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

14.7 Self Assessment Questions and Exercises

1. Prove that the surface generated by the tangents to a twisted curve is a developable surface.
2. Obtain the tangential developable of the curve r=(u, u^2, u^3),
3. Show that the edge of regression of the polar developables of a curve of constant curvature is the locus of its centre of curvature.
4. Find the curvature and torsion of the edge of regression of the osculating developables.
5. Show that the surfce sin z= sinhx. sinhy is minimal.
6. Find the ruled surface formed by the principal normals of a curve.
7. Obtain the distribution of parameter and the striction line of the surface r=(u cosv, u sinv, av).

14.8 Further Readings

1. D.G. Willmore – An Introduction to Differential Geometry, Oxford University Press(1983).
2. D.Somasundaram – Differential Geometry A First Course, Narosa Publishing Pvt.Ltd.(2010).